§5.1 Differentiable Manifolds

Definition A *locally Euclidean space* X of dimension n is a Hausdorff topological space such that, for each $x \in X$, there exists a homeomorphism φ_x mapping some open set containing x onto an open set in $\mathbb{R}^n$.

Remark We may, if we wish, choose each φ_x so that $\varphi_x(x) = 0$ and so that the image of φ_x is a ball $B_0(\varepsilon)$. Given any φ_x homeomorphically mapping an open set U about x onto an open set in $\mathbb{R}^n$, let $\varepsilon > 0$ be such that $B_{\varphi_x(x)}(\varepsilon) \subset \varphi_x(U)$. Let

$$\psi : B_{\varphi_x(x)}(\varepsilon) \to B_0(\varepsilon)$$

be translation by $-\varphi_x(x)$. Then

$$\tilde{\varphi}_x = \psi \circ \varphi_x|_{\varphi_x^{-1}\left(B_{\varphi_x(x)}(\varepsilon)\right)}$$

maps $\varphi_x^{-1}\left(B_{\varphi_x(x)}(\varepsilon)\right)$ homeomorphically onto $B_0(\varepsilon)$.

Example 1. $\mathbb{R}^n$ is locally Euclidean. For each $x \in \mathbb{R}^n$, take φ_x to be the identity map.

Example 2. $\mathbb{S}^n$ is locally Euclidean. Given $x \in \mathbb{S}^n$, let $y \in \mathbb{S}^n$, $y \neq x$. Then φ_x= stereographic projection from y maps $\mathbb{S}^n \setminus \{y\}$ (an open set containing x) homeomorphically onto $\mathbb{R}^n$.

Example 3. Projective space $\mathbb{P}^n$; that is, the space of all lines through 0 in $\mathbb{R}^{n+1}$ is locally Euclidean. For since $\mathbb{P}^n$ is covered by $\mathbb{S}^n$, each $x \in \mathbb{P}^n$ is contained in an open set homeomorphic to an open set in $\mathbb{S}^n$ that itself contains, about each of its points, an open set homeomorphic to an open set $\mathbb{R}^n$.

Example 4. Each open subset U of a locally Euclidean space X is locally Euclidean. For if $x \in U$, let ψ_x be a homeomorphism mapping an open set about x in X onto an open set in $\mathbb{R}^n$. Take $\varphi_x = \psi_x|_{U \cap \text{domain}\, \psi_x}$.

Example 5. Let $M_k(\mathbb{R}) = \{M = (m_{ij})_{1 \le i,j \le k} \mid m_{ij} \in \mathbb{R}, 1 \le i, j \le k\}$ be the space of $k \times k$ matrices with real entries, and let $i : M_k(\mathbb{R}) \to \mathbb{R}^{k^2}$ be the identification map (and a homeomorphism) from $M_k(\mathbb{R})$ onto $\mathbb{R}^{k^2}$ defined by

$$i\,(M) = (m_{11}, \ldots, m_{1k}, m_{21}, \ldots, m_{2k}, \ldots, m_{k1}, \ldots, m_{kk},) \quad \text{for each } M = \begin{pmatrix} m_{11} & \cdots & m_{1k} \\ m_{21} & \cdots & m_{2k} \\ & \cdots & \\ m_{k1} & \cdots & m_{kk} \end{pmatrix}$$

Consider the subset $X = \{M \in M_k(\mathbb{R}) \mid \det M \neq 0\}$ of all nonsingular $k \times k$ matrices. Since the determinant function $\det : M_k(\mathbb{R}) \to \mathbb{R}^1$ defined by

$$\det\,(M) = \sum_{\sigma \in S_k} \text{sgn}\sigma\, m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{k\sigma(k)}, \;\; S_k = \text{permutation group of } k \text{ nuumbers}, \;\; \text{sgn}\sigma = \text{sign of}\, \sigma$$

is continuous on $M_k(\mathbb{R})$ and $\mathbb{R}^1 \setminus \{0\}$ is open in $\mathbb{R}^1$, the set $X = \det^{-1}\left(\mathbb{R}^1 \setminus \{0\}\right)$ is open in $M_k(\mathbb{R})$. Since, for each $M \in X$, the function $\varphi_M = i$ is a coordinate function from the open set X (containing M) in $M_k(\mathbb{R})$ onto the open set $i(X)$ in $\mathbb{R}^{k^2}$, X is a locally Euclidean space.

Definition A *C^k-differentiable manifold* of dimension n is a pair (X, Φ) where X is a Hausdorff topological space, and Φ is a collection of maps such that the following conditions hold (see Figure 5.1) and Φ is called a *C^k-differentiable structure* on X.

(1) $\{\text{domain}\,\varphi\}_{\varphi\in\Phi}$ is an open covering of X,

(2) each $\varphi \in \Phi$ maps its domain homeomorphically onto an open set in $\mathbb{R}^n$,

(3) for each φ, $\psi \in \Phi$ with $(\text{domain}\,\varphi) \cap (\text{domain}\,\psi) \neq \emptyset$, the map $\psi \circ \varphi^{-1}$ is a C^k-map from $\varphi\,(\text{domain}\,\varphi \cap \text{domain}\,\psi) \subset \mathbb{R}^n$ into $\mathbb{R}^n$, that is, any two maps in Φ are C^k-compatible,

(4) Φ is maximal relative to (2) and (3); that is, if

- ψ is any homeomorphism mapping an open set in X onto an open set in $\mathbb{R}^n$
- ψ is C^k-compatible with each map in Φ, that is, for each $\varphi \in \Phi$ with $\text{domain}\,\varphi \cap \text{domain}\,\psi \neq \emptyset$,
  $\psi \circ \varphi^{-1} : \varphi\,(\text{domain}\,\varphi \cap \text{domain}\,\psi) \to \psi\,(\text{domain}\,\varphi \cap \text{domain}\,\psi) \subset \mathbb{R}^n$ and
  $\varphi \circ \psi^{-1} : \psi\,(\text{domain}\,\varphi \cap \text{domain}\,\psi) \to \varphi\,(\text{domain}\,\varphi \cap \text{domain}\,\psi) \subset \mathbb{R}^n$ are C^k-maps,

then $\psi \in \Phi$.

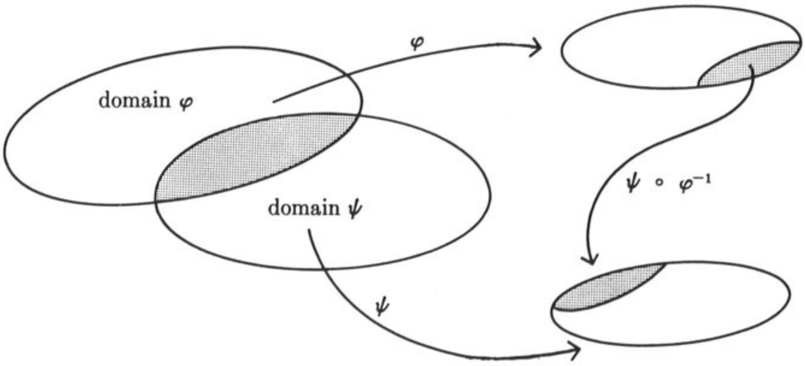


Figure 5.1

Remark Condition (4) implies that if Ψ is a collection of maps satisfying conditions (1), (2), (3) and if $\Phi \cup \Psi$ is a collection of maps satisfying condition (3), then $\Psi \subset \Phi$.

Remark Note that a C^k-manifold is a locally Euclidean space and a locally Euclidean space gives rise to a C^0-manifold.

If $n = 2$ and, in Condition (3), "C^k" is replaced by "complex analytic" (where $\mathbb{R}^2$ is identified with the complex numbers $\mathbb{C}^1$), (X, Φ) is called a *complex analytic manifold* of complex dimension 1 or a Riemann surface. Φ is then called a *complex structure* or *conformal structure* on X.

The maps $\varphi \in \Phi$ are called *coordinate systems*, $(\varphi, \text{domain}\,\varphi)$ are called *coordinate charts* and $\Phi = \{(\varphi, \text{domain}\,\varphi) \mid \varphi \in \Phi\}$ is called a *C^k-atlas* when it satisfies conditions (1) $-$ (3). More precisely, the map $\varphi \in \Phi$ is called *a coordinate system (or chart) on the open set domain$\,\varphi \subset X$*. For $x \in X$, *a coordinate system (or chart) about x* is a coordinate system $\varphi \in \Phi$ such that $x \in \text{domain}\,\varphi$.

Remark Each of the above Examples 1, 2, 3 and 5 of locally Euclidean spaces form the underlying space of a C^∞-manifold. You need only check that the maps φ_x satisfy Condition (3) for a manifold, and then take Φ to be a maximal set containing $\{\varphi_x\}_{x\in X}$. Example 4 above also carries over to manifolds. Namely, if (X, Φ) is a C^k-manifold and U is an open set in X, then $(U, \Phi|_U)$ is a C^k-manifold, where $\Phi|_U = \{\varphi|_U \mid \varphi \in \Phi\}$.

Definition Let (X, Φ) be a C^k-manifold. A real-valued function $f : X \to \mathbb{R}^1$ is a *C^s-function* $(s \le k)$, denoted $f \in C^s(X, \mathbb{R}^1)$, if, for each $\varphi \in \Phi$, $f \circ \varphi^{-1}$ is a C^s-function mapping the image of $\varphi \subset \mathbb{R}^n$ into $\mathbb{R}^1$.

Let (X, Φ) be a C^k-manifold, and let $x \in X$. A real-valued function f is said to be of *class C^s $(s \le k)$, in a neighborhood of x*, denoted $f \in C^s(X, x, \mathbb{R}^1)$, if $U = (\operatorname{domain} f)$ is an open set in X containing x, and $f \in C^s(U, \mathbb{R}^1)$, where U has the C^k-manifold structure as an open set in X.

Remarks Note that we are able to define C^s-functions on X because

(1) X looks locally (via the coordinate systems $\varphi \in \Phi$ like $\mathbb{R}^n$, and we know what it means for a function on $\mathbb{R}^n$ to be C^s;

(2) if $U = \operatorname{domain} \varphi$ and $V = \operatorname{domain} \psi$ for $\varphi, \psi \in \Phi$, with $U \cap V \neq \emptyset$, the concept of a C^s-function in a neighborhood of x in $U \cap V$ is the same relative to the coordinate system φ as to the coordinate system ψ, because $\psi \circ \varphi^{-1}$ is a C^k-homeomorphism and $k \ge s$.

Note also that if f and g are C^s-functions in a neighborhood of x, then $f + g$ and fg (product) are C^s-functions in a neighborhood of x, where

$$\operatorname{domain}(f+g) = \operatorname{domain}(fg) = (\operatorname{domain} f) \cap (\operatorname{domain} g).$$

Definition Let (X, Φ) be a C^k-manifold, and let $\varphi \in \Phi$ be a coordinate system on $U = \operatorname{domain} \varphi$. Let $r_j : \mathbb{R}^n \to \mathbb{R}^1$ be the j^{th} coordinate function on $\mathbb{R}^n$; that is, $r_j(a_1, a_2, \ldots, a_n) = a_j$ for $(a_1, \ldots, a_n) \in \mathbb{R}^n$. The *$j^{th}$ coordinate function* of the coordinate system φ is the function $x_j : U \to \mathbb{R}^1$ defined by $x_j = r_j \circ \varphi$.

Remark $x_j : U \to \mathbb{R}^1$ is a C^k-function. The n-tuple of functions $(x_1, \ldots, x_n)$ is sometimes also referred to as a coordinate system.

Definition Let (X_1, Φ_1) and (X_2, Φ_2) be C^k-manifolds (not necessarily of the same dimension). A mapping $\Psi : X_1 \to X_2$ is of *class C^s $(s \le k)$*, denoted $\Psi \in C^s(X_1, X_2)$, if, whenever $f \in C^s(X_2, \mathbb{R}^1)$, then $f \circ \Psi \in C^s(X_1, \mathbb{R}^1)$.

Exerxise 1. Show that, if $\Psi : X_1 \to X_2$ is of class C^s $(s \ge 0)$, then Ψ is continuous.

Remarks We shall confine our attention to C^∞-manifolds. We shall use the word *"smooth"* to denote C^∞.

We now proceed to define the concept of tangent vector on a manifold. Recall that, in Euclidean space, a vector at a point defines a map which sends each smooth function into a real number, namely, the directional derivative with respect to the given vector. Moreover, the vector is determined by its values on all smooth functions. We shall use this property to define tangent vectors on a manifold.

Definition Let (X, Φ) be a smooth manifold and let $x \in X$. A *tangent vector at x* is a map $v : C^\infty(X, x, \mathbb{R}^1) \to \mathbb{R}^1$ such that, if φ is a (fixed) coordinate system with $x \in U = \operatorname{domain} \varphi$, then there exists an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers with the following property. For each $f \in C^\infty(X, x, \mathbb{R}^1)$,

$$v(f) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i} \left(f \circ \varphi^{-1}\right)\Big|_{\varphi(x)}$$

(Note that if $W = \operatorname{domain} f$, then φ and f both defined on the open set $U \cap W$ containing x, so that $f \circ \varphi^{-1}$ is a smooth function with domain $\varphi(U \cap W) \subset \mathbb{R}^n$ containing $\varphi(x)$.)

Remark If $v : C^\infty(X, x, \mathbb{R}^1) \to \mathbb{R}^1$ has the property required above of a tangent vector with respect to one coordinate system $\varphi = (x_1, \dots, x_n)$; about x, then it also has this property with respect to any other coordinate system about x. For, if $\psi = (y_1, \dots, y_n)$ is another such coordinate system, then, using the chain rule,

$$\begin{aligned}
v(f) &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i} \left(f \circ \varphi^{-1}\right)\Big|_{\varphi(x)} \\
&= \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i} \left(f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}\right)\Big|_{\varphi(x)} \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \frac{\partial}{\partial r_j} \left(f \circ \psi^{-1}\right)\Big|_{\psi(x)} J_{ji}\left(\psi \circ \varphi^{-1}\right)\big|_{\varphi(x)} \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_i\, J_{ji}\left(\psi \circ \varphi^{-1}\right)\big|_{\varphi(x)} \right) \frac{\partial}{\partial r_j} \left(f \circ \psi^{-1}\right)\Big|_{\psi(x)} \\
&= \sum_{j=1}^{n} b_j \frac{\partial}{\partial r_j} \left(f \circ \psi^{-1}\right)\Big|_{\psi(x)}
\end{aligned}$$

where $J_{ji}\left(\psi \circ \varphi^{-1}\right) = \partial y_j / \partial x_i$ is the Jacobian matrix of the function $\psi \circ \varphi^{-1}$, and $b_j = \sum_{i=1}^{n} a_i\, J_{ji}\left(\psi \circ \varphi^{-1}\right)\big|_{\varphi(x)}$. Thus, to check if v is a tangent vector at x, it suffices to check the required property in any one coordinate system at x.

Notation Given a coordinate system φ about x, let $x_j = r_j \circ \varphi$ denote the j^{th} coordinate function of φ. By $\partial/\partial x_j$ $(j = 1, \dots, n)$ is meant the tangent vector at x defined by

$$\frac{\partial}{\partial x_j}(f) = \frac{\partial}{\partial r_j}\left(f \circ \psi^{-1}\right)\Big|_{\psi(x)} \qquad \text{for } f \in C^\infty(X, x, \mathbb{R}^1)$$

Thus $\dfrac{\partial}{\partial x_j}$ corresponds, relative to the coordinate system φ, to the n-tuple $(0, 0, \dots, 1, \dots, 0)$, where the 1 is in the j^{th} spot.

Remark 1. If $x_1, \dots, x_n$ are the coordinate functions of a coordinate system φ about x, and $y_1, \dots, y_n$ are those of a coordinate system ψ about x, then the above computation shows that

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_j} \frac{\partial}{\partial y_i} \quad \text{for each } j = 1, \dots, n$$

Remark 2. A tangent vector v at $x \in X$ has the following properties. For any $f, g \in C^\infty(X, x, \mathbb{R}^1)$ and for $\lambda \in \mathbb{R}^1$,

(1) $v(f + g) = v(f) + v(g)$,

(2) $v(\lambda f) = \lambda v(f)$,

(3) $v(fg) = v(f)g(x) + f(x)v(g)$.

These three properties say that the map $v : C^\infty(X, x, \mathbb{R}^1) \to \mathbb{R}^1$ is a *derivation*.

Moreover, these properties characterize tangent vectors; that is, we could have defined a tangent vector to be a map $v : C^\infty(X, x, \mathbb{R}^1) \to \mathbb{R}^1$ satisfying $(1) - (3)$ above, and then proved that, relative to any coordinate system φ about x, $v = \sum_{i=1}^{n} a_i\,(\partial/\partial x_i)$ for some n-tuple $(a_1, \dots, a_n)$ of real numbers, where x_i is the i^{th} coordinate function of φ.

Remark 3. The set X_x of tangent vectors at x forms a vector space under the following rules of addition and scalar multiplication:

$$\begin{aligned}(v_1 + v_2)(f) &= v_1(f) + v_2(f) \quad \text{for all } v_1, v_2 \in X_x \\ (\lambda v_1)(f) &= \lambda v_1(f) \quad \text{for all } v_1, \in X_x,\ \lambda \in \mathbb{R}^1.\end{aligned}$$

To see that $v_1 + v_2$ and λv_1 are tangent vectors at x, let φ be a coordinate system about x, with coordinate functions $(x_1, \dots, x_n)$. Then

$$v_1 = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad v_2 = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$

for some $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$. It is then easy to check that the following rules of addition and scalar multiplication:

$$\begin{aligned}v_1 + v_2 &= \sum_{i=1}^{n} (a_i + b_i) \frac{\partial}{\partial x_i} \\ \lambda v_1 &= \sum_{i=1}^{n} (\lambda a_i) \frac{\partial}{\partial x_i}\end{aligned}$$

The map $(a_1, \dots, a_n) \to \sum_{i=1}^{n} a_i\,(\partial/\partial x_i)$ gives a vector space isomorphism $\mathbb{R}^n \to X_x$, so X_x has dimension n. Moreover, it is clear that $\{\partial/\partial x_i \mid i = 1, \dots, n\}$ is a basis for X_x. The space X_x is called the *tangent space* to X at x. It is also denoted by $T(X)_x$ or by $T(X, x)$.

For φ and ψ two coordinate systems at x, with coordinate functions $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$, respectively, the formula

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_j} \frac{\partial}{\partial y_i} \quad \text{for each } j = 1, \dots, n$$

merely expresses the vector $\partial/\partial x_j$ in terms of the basis $\{\partial/\partial y_i \mid i = 1, \dots, n\}$. Thus the change of basis matrix from the basis $\{\partial/\partial y_i \mid i = 1, \dots, n\}$. of X_x to the basis $\{\partial/\partial x_i \mid i = 1, \dots, n\}$ is precisely the Jacobian matrix $\left(\dfrac{\partial y_i}{\partial x_j}\right)_{1 \le i, j \le n}$.

Remark 4. The tangent space $T(\mathbb{R}^n, a)$ to $\mathbb{R}^n$ at a point $a \in \mathbb{R}^n$ is naturally isomorphic with $\mathbb{R}^n$ itself. The isomorphism $\mathbb{R}^n \to T(\mathbb{R}^n, a)$ is given by

$$(\lambda_1, \dots, \lambda_n) \to \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial r_i}$$

Notation We shall henceforth omit the Φ from our notation for a differentiable manifold (X, Φ). To be sure, a locally Euclidean space X may have two or more distinct differentiable structures on it (or it may have none), but we shall denote a manifold (X, Φ) merely by X and shall assume that a definite differentiable structure is given on it.

Definition Let X and Y be smooth manifolds. Let $\Psi : X \to Y$ be a smooth map. The *differential* of Ψ at $x \in X$ is the map $d\Psi : X_x \to Y_{\Psi(x)}$ defined as follows. For $v \in X_x$ and $g \in C^\infty(Y, \Psi(x), \mathbb{R}^1)$,

$$(d\Psi(v))(g) = v(g \circ \Psi)$$

Remark It is easily checked that $d\Psi(v)$is indeed a tangent vector at $Y_{\Psi(x)}$. For, if φ is a coordinate system about x with coordinate functions $(x_1, \ldots, x_n)$, and τ is a coordinate system about $\Psi(x)$ with coordinate functions $(y_1, \ldots, y_m)$, then

$$\begin{aligned}
[d\Psi(v)](g) &= v(g \circ \Psi) = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}(g \circ \Psi) \\
&= \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\left(g \circ \tau^{-1} \circ \tau \circ \Psi \circ \varphi^{-1}\right)\Big|_{\varphi(x)} \\
&= \sum_{i=1}^n a_i \sum_{j=1}^m \frac{\partial}{\partial s_j}\left(g \circ \tau^{-1}\right)\Big|_{\tau \circ \Psi(x)} \frac{\partial}{\partial r_i}\left(s_j \circ \tau \circ \Psi \circ \varphi^{-1}\right)\Big|_{\varphi(x)} \\
&\qquad \text{where } (s_1, \ldots, s_m) \text{ are coordinates on } \mathbb{R}^m \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i \frac{\partial}{\partial y_j}(g) \frac{\partial}{\partial x_i}(y_j \circ \Psi) \\
&= \left[\sum_{j=1}^m v(y_j \circ \Psi) \frac{\partial}{\partial y_j}\right](g)
\end{aligned}$$

Since this holds for all $g \in C^\infty(Y, \Psi(x), \mathbb{R}^1)$,

$$\boxed{d\Psi(v) = \sum_{j=1}^m v(y_j \circ \Psi) \frac{\partial}{\partial y_j}}$$

and, in particular, $d\Psi(v)$ is a tangent vector. Furthermore, it is clear that $d\Psi$ is a linear transformation $X_x \to Y_{\Psi(x)}$. Since

$$d\Psi\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^m \frac{\partial}{\partial x_i}(y_j \circ \Psi) \frac{\partial}{\partial y_j}$$

this linear transforination $d\Psi$ has matrix

$$(d\Psi)_{ij} = \left(\frac{\partial}{\partial x_j}(y_i \circ \Psi)\right)$$

relative to the bases $\{\partial/\partial x_i \mid i = 1, \ldots, n\}$ and $\{\partial/\partial y_j \mid j = 1, \ldots, m\}$.

Remark Let X, Y and Z be smooth manifolds. Let $\Psi : X \to Y$ and $\Phi : Y \to Z$ be smooth maps. Then $d(\Phi \circ \Psi) = d\Phi \circ d\Psi$.

Proof Suppose $v \in X_x$ and $h \in C^\infty(Z, \Phi \circ \Psi(x), \mathbb{R}^1)$. Then

$$\begin{aligned}
[d(\Phi \circ \Psi)(v)](h) &= v(h \circ (\Phi \circ \Psi)) = v(h \circ \Phi) \circ \Psi) \\
&= d\Psi(v)(h \circ \Phi) \\
&= [d\Phi(d\Psi(v))](h) \\
&= [(d\Phi \circ d\Psi)(v)](h).
\end{aligned}$$

Remark Let X be a smooth manifold, and let U be open in X. Then U is itself a smooth manifold. Moreover, the inclusion map $i : U \to X$ is a smooth map. Indeed, $f \in C^\infty(X, \mathbb{R}^1)$ implies $f|_U \in C^\infty(U, \mathbb{R}^1)$. Furthermore, the differential

$$di : T(U, u_0) \to T(x, u_0) \quad \text{for each } u_0 \in U$$

is an isomorphism; we shall identify these two linear spaces.

Exercise 2. If U is an open set in X and $u_0 \in U$, construct a function $h \in C^\infty(X, \mathbb{R}^1)$ such that

$$h(x) = \begin{cases} 1 & \text{if } x \in W \text{ an open set containing } u_0 \\ 0 & \text{if } x \notin U \end{cases}$$

Hint: Make use of the smooth function $g : \mathbb{R}^1 \to \mathbb{R}^1$ defined by

$$g(t) = \begin{cases} e^{-1/t^2} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

If $f_1 \in C^\infty(U, u_0, \mathbb{R}^1)$, use Exercise 1 to show that there exists a smaller open set W and $f \in C^\infty(X, \mathbb{R}^1)$ such that $f|_W = f_1|_W$.

Remark Let X be a smooth manifold, and let $f \in C^\infty(X, \mathbb{R}^1)$. Let us compute df. For $v \in T(X, x)$, $df(v) \in T(\mathbb{R}^1, f(x))$. Since $T(\mathbb{R}^1, f(x))$ is 1-dimensional, $df(v) = \lambda\,(d/dr)$ for some $\lambda \in \mathbb{R}^1$. To determine λ, it suffices to evaluate $df(v)$ on the coordinate function $r : \mathbb{R}^1 \to \mathbb{R}^1$ as follows.

$$\lambda = \left[\lambda \frac{d}{dr}\right](r) = [df(v)]\,(r) = v(r \circ f) = v(f) \implies df(v) = v(f)\frac{d}{dr}$$

Now $T(\mathbb{R}^1, f(x))$ is naturally isomorphic with $\mathbb{R}^1$ via the isomorphism

$$\lambda \frac{d}{dr} \to \lambda \quad \text{for eaach } \lambda \in \mathbb{R}^1$$

Let us identify these two spaces through this isomorphism. Then $df : T(X, x) \to \mathbb{R}^1$ is a *linear functional* on $T(X, x)$; that is, df is a member of the *dual space* $T^*(X, x)$ and is, as such, given by

$$df(v) = v(f) \quad \text{for each } v \in T(X, x)$$

$T^*(X, x)$ is called the *cotangent space* at x.

Definition Let X be a smooth manifold. A *smooth curve* in X is a smooth map α from some (open or closed) interval $I \subset \mathbb{R}^1$ into X. If the domain of α is a closed interval $[a, b]$, smoothness of α means that α admits a smooth *extension*

$$\tilde{\alpha} : (a - \varepsilon, b + \varepsilon) \to X.$$

(Note that open intervals are open sets in $\mathbb{R}^1$ and hence are smooth manifolds.)

A *broken C^∞-curve* in X is a continuous map $\alpha : [a, b] \to X$ together with a subdivision of $[a, b]$ on whose closed subintervals α is a C^∞ curve.

Example

$$\alpha(t) = \begin{cases} (t, t\sin 1/t) & \text{if } t \in (0, 1] \\ (0, 0) & \text{if } t = 0 \end{cases}$$

is not a smooth curve in $\mathbb{R}^2$ because it admits no smooth extension past 0.

Definition Let I be an interval in $\mathbb{R}^1$, $\alpha : I \to X$ be a smooth curve in X. The *tangent vector* to α at time t ($t \in I$), denoted by $\dot{\alpha}(t)$, is defined by

$$\dot{\alpha}(t) = d\tilde{\alpha}\left(\left(\frac{d}{dr}\right)_t\right)$$

Note that $\dot{\alpha}(t)$ is well defined, even at the endpoints of I.

Remark Given a tangent vector $v \in X_x$ let $\alpha : I \to X$ be a smooth curve whose tangent vector at time $t = 0$ is v (Such a curve may be obtained by taking a coordinate system φ about x, finding a curve (for example, the straight line) in $\mathbb{R}^n$ whose tangent vector at time 0 is $d\varphi(v)$ and pulling this curve back to X by φ^{-1}.) Then, for $f \in C^\infty(X, x, \mathbb{R}^1)$,

$$v(f) = \dot{\alpha}(0)(f) = d\tilde{\alpha}\left(\left(\frac{d}{dr}\right)_0\right)(f) = \frac{d}{dr}(f \circ \tilde{\alpha})\bigg|_0$$

Thus $v(f)$ is the derivative of the "restriction" of f to the curve α. Moreover, two curves α_1 and α_2 have the same tangent vector v at time 0 if and only if $\alpha_1(0) = \alpha_2(0)$ and

$$\frac{d}{dr}(f \circ \tilde{\alpha}_1)\bigg|_0 = \frac{d}{dr}(f \circ \tilde{\alpha}_2)\bigg|_0 \quad \text{for all } f \in C^\infty(X, x\mathbb{R}^1) \text{ (see Figure 5.2)}$$

We may use this equation to define an equivalence relation on the set of all curves α with $\alpha(0) = x$. Then we get a one-to-one correspondence between equivalence classes of curves through x and tangent vectors at x. Thus, we could have defined a tangent vector at x to be such an equivalence class of curves through x.

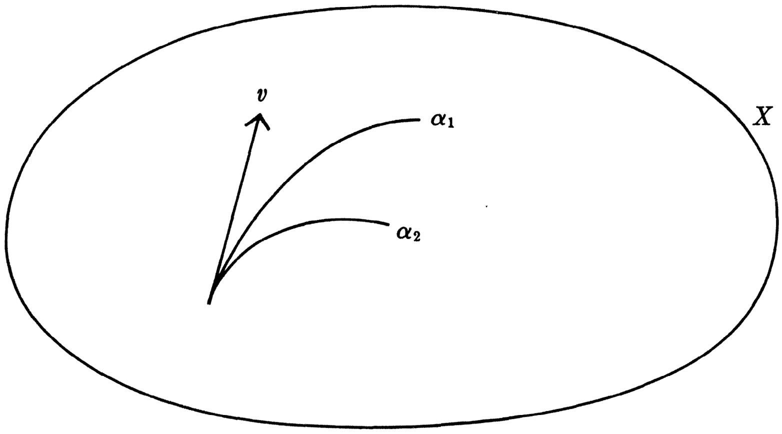


Figure 5.2

§5.2 Differential Forms

Let X be a smooth manifold. Define (see Figure 5.3)

$$T(X) = \bigcup_{x \in X} T(X, x) \quad \text{and} \quad T^*(X) = \bigcup_{x \in X} T^*(X, x)$$

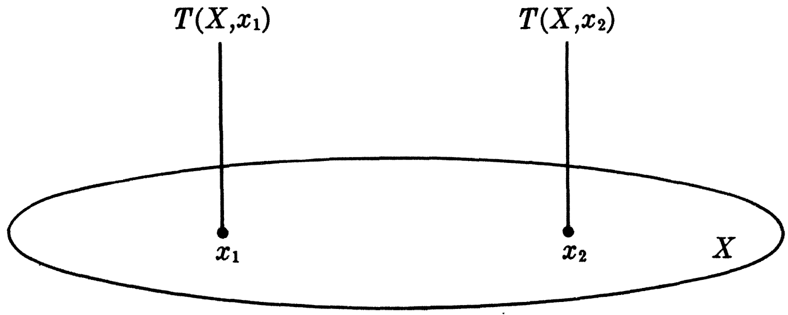


Figure 5.3

$T(X)$ is called the *tangent bundle* of X. $T^*(X)$ is called the *cotangent bundle* of X.

A *projection map* $\pi : T(X) \to X$ is defined as follows. If $v \in T(X)$, then $v \in T(X, x)$ for some (unique) $x \in X$; set $\pi(v) = x$. Similarly, there is a projection map from $T^*(X)$ onto X that we shall also denote by π.

A *vector field* on X is a map $V : X \to T(X)$ such that $\pi \circ V = i_X$. A vector field V is *smooth* if for each $f \in C^\infty(X, \mathbb{R}^1)$, $Vf \in C^\infty(X, \mathbb{R}^1)$. Here Vf is defined by

$$(Vf)(x) = V(x)f$$

A *differential 1-form* on X is a map $\omega : X \to T^*(X)$ such that $\pi \circ \omega = i_X$. A differential 1-form ω is *smooth* if for each smooth vector field V on X,

$$\omega(V) \in C^\infty(X, \mathbb{R}^1).$$

Here $\omega(V)$ is defined by $(\omega(V))(x) = \omega(x)(V(x))$. We shall denote the set of all smooth vector fields on X by $C^\infty(X, T(X))$ and the set of all smooth 1-forms by $C^\infty(X, T^*(X))$.

Exercise 3. Define a manifold structure on $T(X)$ so that π is a smooth map and so that a vector field V is smooth if and only if it is a smooth map from $X \to TX$.

Hint: For $\varphi : U \to \mathbb{R}^n$ a local coordinate system on X, with coordinate functions $(x_1, \ldots, x_n)$, define $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\tilde{\varphi}(v) = (\varphi \circ \pi(v), b_1, \ldots, b_n), \quad \text{where } b_1, \ldots, b_n \in \mathbb{R}^1 \text{ are such that } v = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$

Remark 1. Let $f \in C^\infty(X, \mathbb{R}^1)$. Then $df \in C^\infty(X, T^*(X))$. For if $V \in C^\infty(X, T(X))$, then $df(V) = Vf \in C^\infty(X, \mathbb{R}^1)$.

Remark 2. $C^\infty(X, T(X))$ and $C^\infty(X, T^*(X))$ are both vector spaces over the reals under the operations of pointwise addition and scalar multiplication. For example, if V_1, $V_2 \in C^\infty(X, T(X))$ and if $\lambda \in \mathbb{R}^1$, then $V_1 + V_2$ and λV_1 are defined by

$$(V_1 + V_2)(x) = V_1(x) + V_2(x) \quad \text{and} \quad (\lambda V_1)(x) = \lambda(V_1(x)) \quad \text{for each } x \in X.$$

Remark 3. Let φ be a coordinate system on X with domain U and coordinate functions $(x_1, x_2, \ldots, x_n)$. Then the following hold.

(1) $(\partial/\partial x_i) \in C^\infty(U, T(U))$ for $i \in \{1, \ldots, n\}$. $\partial/\partial x_i$ is smooth because if

$$f \in C^\infty(U, \mathbb{R}^1), \quad \text{then } f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^1),$$

and, for each $x \in U$,

$$\begin{aligned} \left[\frac{\partial}{\partial x_i}(f)\right] &= \left[\frac{\partial}{\partial r_i}(f \circ \varphi^{-1})\right](\varphi(x)) \\ &= \left[\left[\frac{\partial}{\partial r_i}(f \circ \varphi^{-1})\right] \circ \varphi\right](x) \end{aligned}$$

that is,

$$\frac{\partial}{\partial x_i}(f) = \left[\frac{\partial}{\partial r_i}(f \circ \varphi^{-1})\right] \circ \varphi \in C^\infty(U, \mathbb{R}^1).$$

(2) If $V \in C^\infty(U, T(U))$, then there exist functions $a_i \in C^\infty(U, \mathbb{R}^1)$, for $i \in \{1, \ldots, n\}$, such that

$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$

These functions a_i exist because $\{(\partial/\partial x_i)(x) \mid 1 \leq i \leq n\}$ is a basis for $T(X, x)$. They are smooth because $(\partial/\partial x_i)(x_j) = \delta_{ij}$, so that

$$a_j = \sum_{i=1}^{n} a_i \delta_{ij} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}(x_j) = V(x_j) \in C^\infty(U, \mathbb{R}^1).$$

(3) If $V \in C^\infty(X, T(X))$, then $V|_U \in C^\infty(U, T(U))$ by the previous exercise, and $V|_U = \sum_{i=1}^{n} a_i(\partial/\partial x_i)$ as in (2) with $a_i \in C^\infty(U, \mathbb{R}^1)$.

(4) $dx_j \in C^\infty(U, T^*(U))$ for $j \in \{1, \ldots, n\}$ because $x_j \in C^\infty(U, \mathbb{R}^1)$. Furthermore, $\{dx_j\}$ is at each point the dual basis to $\{\partial/\partial x_j\}$ because

$$dx_j\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}(x_j) = \delta_{ij}.$$

(5) If $\omega \in C^\infty(X, T^*(X))$, then there exist $a_i \in C^\infty(U, \mathbb{R}^1)$ such that $\omega = \sum_{i=1}^{n} a_i dx_i$. These functions $\{a_i \mid 1 \leq i \leq n\}$ exist because $\{dx_i\}$ is at each point a basis for the cotangent space. They are smooth because

$$a_i = \sum_{j=1}^{n} a_j dx_j\left(\frac{\partial}{\partial x_i}\right) = \omega\left(\frac{\partial}{\partial x_i}\right) \in C^\infty(U, \mathbb{R}^1).$$

(6) If $f \in C^\infty(U, \mathbb{R}^1)$, then

$$df = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(f) dx_j$$

because $df = \sum_{j=1}^{n} a_j dx_j$ for some $\{a_j \mid 1 \le j \le n\}$, and

$$a_i = \sum_{j=1}^{n} a_j dx_j \left(\frac{\partial}{\partial x_j}\right) = df\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_j}(f)$$

We have just seen that if $f \in C^\infty(U, \mathbb{R}^1)$, then df is a smooth differential 1-form. We now introduce differential k-forms.

Review of Exterior Algebra Let V be an n-dimensional vector space over the reals and let V^k denote the k-fold product $V \times \cdots \times V$.

Definition A function $\tau : V^k \to \mathbb{R}$ is called *multilinear* if for each j with $1 \le j \le k$, for each $v_1, \ldots, v_k$, $v_j' \in V$ and $\lambda \in \mathbb{R}^1$, we have

$$\begin{aligned}
\tau(v_1, \ldots, v_{j-1}, v_j + v_j', v_{j+1}, \ldots, v_k) &= \tau(v_1, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_k) \\
&\quad + \tau(v_1, \ldots, v_{j-1}, v_j', v_{j+1}, \ldots, v_k), \\
\tau(v_1, \ldots, v_{j-1}, \lambda v_j, v_{j+1}, \ldots, v_k) &= \lambda\, \tau(v_1, \ldots, v_j, \ldots, v_k).
\end{aligned}$$

A multilinear function $\tau : V^k \to \mathbb{R}$ is called a *k-tensor* on V, and the set of all k-tensors, denoted $\mathscr{T}^k(V^*)$, becomes a vector space (over $\mathbb{R}$) if for each τ, $\eta \in \mathscr{T}^k(V^*)$, $v_1, \ldots, v_k \in V$ and $\lambda \in \mathbb{R}^1$, we define

$$\begin{aligned}
(\tau + \eta)(v_1, \ldots, v_k) &= \tau(v_1, \ldots, v_k) + \eta(v_1, \ldots, v_k) \\
(\lambda\tau)(v_1, \ldots, v_k) &= \lambda\tau(v_1, \ldots, v_k)
\end{aligned}$$

There is also an operation connecting the various spaces $\mathscr{T}^k(V^*)$. If $\tau \in \mathscr{T}^k(V^*)$ and $\eta \in \mathscr{T}^\ell(V^*)$, we define the *tensor product $\tau \otimes \eta \in \mathscr{T}^{k+\ell}(V^*)$* by

$$\tau \otimes \eta\,(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = \tau(v_1, \ldots, v_k) \cdot \eta(v_{k+1}, \ldots, v_{k+\ell}) \quad \text{for all } v_1, \ldots, v_{k+\ell} \in V.$$

Note that the order of the factors τ and η is crucial here since $\tau \otimes \eta$ and $\eta \otimes \tau$ are far from equal.

Exercise 4. Use the definition to show that if S, $S_1, S_2 \in \mathscr{T}^k(V^*)$, T, $T_1, T_2 \in \mathscr{T}^\ell(V^*)$, $U \in \mathscr{T}^m(V^*)$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned}
(S_1 + S_2) \otimes T &= S_1 \otimes T + S_2 \otimes T \\
S \otimes (T_1 + T_2) &= S \otimes T_1 + S \otimes T_2 \\
(\lambda S) \otimes T &= S \otimes (\lambda T) = \lambda\,(S \otimes T) \\
(S \otimes T) \otimes U &= S \otimes (T \otimes U)
\end{aligned}$$

Both $(S \otimes T) \otimes U$ and $S \otimes (T \otimes U)$ are usually denoted simply $S \otimes T \otimes U$; higher-order products $T_1 \otimes \cdots \otimes T_r$ are defined similarly. Note that $\mathscr{T}^1(V^*)$ is the dual space V^* and the operation $\otimes$ allows us to express $\mathscr{T}^k(V^*)$ in terms of $\mathscr{T}^1(V^*) \cong V^*$.

Theorem Let $\{v_i\}_{i=1}^n$ be a basis for V, and let $\{\varphi_i \in V^* \mid 1 \le i \le n\}$ be the dual basis, i.e.

$$\varphi_i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

Then the set of all k-fold tensor products

$$\mathscr{B} = \{\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \mid 1 \le i_1, \ldots, i_k \le n\}$$

is a basis for $\mathscr{T}^k(V^*)$, which therfore has dimension n^k.

Proof Since

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} (v_{j_1}, \ldots, v_{j_k}) = \delta i_i j_1 \cdots \delta_{i_k j_k} = \begin{cases} 1 & \text{if } i_1 = j_1, \ldots, i_k = j_k \\ 0 & \text{otherwise.} \end{cases}$$

for any $T \in \mathscr{T}^k(V^*)$, and for any vectors $w_1, \ldots, w_k \in V$ with $w_i = \sum_{j=1}^{n} a_{ij} v_j$, we have $\varphi_\ell(w_i) = a_{i\ell}$ and

$$\begin{aligned} T(w_1, \ldots, w_k) &= \sum_{j_1,\ldots,j_k=1}^{n} a_{1,j_1} \cdots a_{k,j_k} T(v_{j_1}, \ldots, v_{j_k}) \\ &= \sum_{j_1,\ldots,j_k=1}^{n} T(v_{j_1}, \ldots, v_{j_k}) \cdot \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k} (w_{j_1}, \ldots, w_{j_k}) . \end{aligned}$$

Thus

$$T = \sum_{i_1,\ldots,i_k=1}^{n} T(v_{i_1}, \ldots, v_{i_k}) \cdot \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k},$$

that is, $\mathscr{B} = \{\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \mid 1 \le i_1, \ldots, i_k \le n\}$ spans $\mathscr{T}^k(V^*)$.

Suppose now that there are numbers $a_{i_1 \ldots i_k}$ such that

$$\sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\ldots i_k} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} = 0 \implies a_{j_1\ldots j_k} = \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\ldots i_k} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(v_{j_1}, \ldots, v_{j_k}) = 0$$

Thus $\{\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \mid 1 \le i_1, \ldots, i_k \le n\}$ are linearly independent.

Remark Let V and W be vector spaces over $\mathbb{R}$. If $f : V \to W$ is a linear transformation, then a linear transformation $f^* : \mathscr{T}^k(W^*) \to \mathscr{T}^k(V^*)$ is defined by

$$f^*T\,(v_1, \ldots, v_k) = T\,(f(v_1), \ldots, f(v_k)) \quad \text{for each } T \in \mathscr{T}^k(V^*) \text{ and } v_1, \ldots, v_k \in V.$$

Exercise 5. Show that

$$f^*\,(S \otimes T) = f^*S \otimes f^*T \quad \text{for all } S,\, T \in \mathscr{T}^k(V^*).$$

Example Let $\langle\, ,\, \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the usual inner product on $\mathbb{R}^n$. Note that $\langle\, ,\, \rangle \in \mathscr{T}^2(\mathbb{R}^n)$, $\langle v, w\rangle = \langle w, v\rangle$ for $v,\, w \in \mathbb{R}^n$ and $\langle v, v\rangle > 0$ if $v \ne 0$.

In general, if V is an n-dimensional vector space over $\mathbb{R}$, we define an *inner product* on V to be a 2-tensor T such that T is *symmetric*, that is $T(v, w) = T(w, v)$ for $v,\, w \in V$ and such that T is *positive definite*, that is, $T(v, v) > 0$ if $v \ne 0$.

Theorem If T is an inner product on V, there is a basis $\{v_i \mid 1 \le i \le n\}$ for V such that $T(v_i, v_j) = \delta_{ij}$. (Such a basis is called *orthonormal* with respect to T.) Consequently, there is an isomorphism $f : \mathbb{R}^n \to V$ such that

$$T(f(x), f(y)) = \langle x, y\rangle \quad \text{for } x,\, y \in \mathbb{R}^n \iff f^*T = \langle\, ,\, \rangle.$$

Proof Let $w_1, \ldots, w_n$ be any basis for V. Use the Gram-Schmidt process to define

$$\begin{aligned}
w_1' &= w_1 \neq 0 \implies T(w_1', w_1') > 0, \\
w_2' &= w_2 - \frac{T(w_1', w_2)}{T(w_1', w_1')} \cdot w_1' \neq 0 \implies T(w_2', w_2') > 0,\, T(w_1', w_2') = 0, \\
w_3' &= w_3 - \sum_{\ell=1}^{2} \frac{T(w_\ell', w_3)}{T(w_\ell', w_\ell')} \cdot w_\ell' \neq 0 \implies T(w_3', w_3') > 0,\, T(w_\ell', w_3') = 0 \text{ for } 1 \leq \ell \leq 2, \\
\cdots &\quad \cdots \quad \cdots \\
w_j' &= w_j - \sum_{\ell=1}^{j-1} \frac{T(w_\ell', w_j)}{T(w_\ell', w_\ell')} \cdot w_\ell' \neq 0 \implies T(w_j', w_j') > 0,\, T(w_\ell', w_j') = 0 \text{ for } 1 \leq \ell \leq j-1, \\
\cdots &\quad \cdots \quad \cdots
\end{aligned}$$

Since $\{w_i\}$ is a basis for V, $w_i' \neq 0 \implies T(w_i', w_i') > 0$ for each $1 \leq i \leq n$. Also it is easy to check that if $i \neq j$, then $T(w_i', w_j') = 0$, i.e. $\{w_i'\}$ is an orthogonal basis for V. Now define

$$v_i = \frac{w_i'}{\sqrt{T(w_i', w_i')}} \implies \{v_i\}_{i=1}^n \text{ is an orthonormal basis with respect to } T.$$

The isomorphism f may be defined by $f(e_i) = v_i$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$.

Definition A k-tensor $\tau \in \mathscr{T}^k(V^*)$ is called *alternating (or skew-symmetric)* on V if for all $v_1, \ldots, v_k \in V$, and for any $1 \leq i < j \leq k$,

$$\tau(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

Let $\Lambda^k(V^*) = \{\tau \in \mathscr{T}^k(V^*) \mid \tau(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)\}$ denote the set of all alternating k-tensors. Then it is clear that $\Lambda^k(V^*)$ is a subspace of $\mathscr{T}^k(V^*)$.

For each $\tau \in \mathscr{T}^k(V^*)$ and for all $v_1, \ldots, v_k \in V$, we define $\mathrm{Alt}\,(\tau)$ by

$$\mathrm{Alt}\,(\tau)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma\, \tau(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \left[\overset{\text{or}}{=} \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma\, \tau(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \right].$$

Exercise 6. Show that if $\tau \in \Lambda^k(V^*)$, $v_1 \ldots, v_k \in V$ and $v_i = v_j$ for some $1 \leq i < j \leq k$, then

$$\tau(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = 0$$

Theorem

(1) If $\tau \in \mathscr{T}^k(V^*)$, then $\mathrm{Alt}\,(\tau) \in \Lambda^k(V^*)$.

(2) If $\omega \in \Lambda^k(V^*)$, then $\mathrm{Alt}\,(\omega) = \omega$, i.e. $\mathrm{Alt}\,|_{\Lambda^k(V^*)} = \mathrm{id}\,|_{\Lambda^k(V^*)}$.

(3) If $\tau \in \mathscr{T}^k(V^*)$, then $\mathrm{Alt}\,(\mathrm{Alt}\,(\tau)) = \mathrm{Alt}\,(\tau)$.

Proof For $1 \leq i < j \leq n$, let $\sigma_0 = (i, j) \in S_k$ be the permutation that interchanges i and j and leaves all other numbers fixed, i.e.

$$\sigma_0(\ell) = \begin{cases} \ell & \text{if } \ell \neq i,\, j \\ j & \text{if } \ell = i \\ i & \text{if } \ell = j \end{cases}$$

(1) For each $\sigma \in S_k$, since

$$\sigma_0 \cdot \sigma_0 = \mathrm{id} \implies \sigma_0^{-1} = \sigma_0,$$

there is a unique $\sigma' = \sigma \cdot \sigma_0 \in S_k$ such that $\sigma = \sigma' \cdot \sigma_0$.

$$\begin{aligned}
& \mathrm{Alt}\,(\tau)\,(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k) \\
= \; & \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\,\sigma\, \tau(v_{\sigma(1)}, \ldots, v_{\sigma(j)}, \ldots, v_{\sigma(i)}, \ldots, v_{\sigma(k)}) \\
= \; & \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\,\sigma\, \tau(v_{\sigma'\cdot\sigma_0(1)}, \ldots, v_{\sigma'\cdot\sigma_0(j)}, \ldots, v_{\sigma'\cdot\sigma_0(i)}, \ldots, v_{\sigma'\cdot\sigma_0(k)}) \\
= \; & \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\,\sigma\, \tau(v_{\sigma'(1)}, \ldots, v_{\sigma'(i)}, \ldots, v_{\sigma'(j)}, \ldots, v_{\sigma'(k)}) \\
= \; & \frac{1}{k!} \sum_{\sigma' \in S_k} \mathrm{sgn}\,(\sigma' \cdot \sigma_0)\, \tau(v_{\sigma'(1)}, \ldots, v_{\sigma'(i)}, \ldots, v_{\sigma'(j)}, \ldots, v_{\sigma'(k)}) \\
= \; & \frac{1}{k!} \sum_{\sigma' \in S_k} -\mathrm{sgn}\,\sigma'\, \tau(v_{\sigma'(1)}, \ldots, v_{\sigma'(i)}, \ldots, v_{\sigma'(j)}, \ldots, v_{\sigma'(k)}) \\
= \; & -\mathrm{Alt}\,(\tau)\,(v_1, \ldots, v_k)
\end{aligned}$$

(2) If $\omega \in \Lambda^k(V^*)$, $\sigma_0 = (i,j) \in S_k$ and $v_1, \ldots, v_k \in V$, then

$$\omega\left(v_{\sigma_0(1)}, \ldots, v_{\sigma_0(k)}\right) = \mathrm{sgn}\,\sigma_0 \cdot \omega\left(v_1, \ldots, v_k\right).$$

Since every $\sigma \in S_k$ is a product of permutations of the form (i,j), we have

$$\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = \mathrm{sgn}\,\sigma \cdot \omega\left(v_1, \ldots, v_k\right) \quad \text{for all } \sigma \in S_k.$$

Therefore

$$\begin{aligned}
\mathrm{Alt}\,(\omega)\,(v_1, \ldots, v_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\,\sigma\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \\
&= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\,\sigma \cdot \mathrm{sgn}\,\sigma\, \omega(v_1, \ldots, v_k) \\
&= \omega(v_1, \ldots, v_k)
\end{aligned}$$

(3) follows immediately from (1) and (2).

Definition Let $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^\ell(V^*)$. Define the *wedge product* $\omega \wedge \eta \in \Lambda^{k+\ell}(V^*)$ by

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\,\ell!}\,\mathrm{Alt}\,(\omega \otimes \eta).$$

Exercise 7. Let V and W be vector spaces over $\mathbb{R}$, $f : V \to W$ be a linear transformation, and let ω, ω_1, $\omega_2 \in \Lambda^k(W^*)$, η, η_1, $\eta_2 \in\in \Lambda^\ell(W^*)$ and $a \in \mathbb{R}$. Show that the following equations hold.

$$\begin{aligned}
(\omega_1 + \omega_2) \wedge \eta &= \omega_1 \wedge \eta + \omega_2 \wedge \eta, \\
\omega \wedge (\eta_1 + \eta_2) &= \omega \wedge \eta_1 + \omega \wedge \eta_2, \\
a\omega \wedge \eta &= \omega \wedge a\eta = a\,(\omega \wedge \eta), \\
\omega \wedge \eta &= (-1)^{k\ell}\, \eta \wedge \omega, \\
f^*\,(\omega \wedge \eta) &= f^*\,(\omega) \wedge f^*\,(\eta).
\end{aligned}$$

Theorem

(1) If $S \in \mathscr{T}^k(V^*)$, $T \in \mathscr{T}^\ell(V^*)$ and $\operatorname{Alt}(S) = 0$ then

$$\operatorname{Alt}(S \otimes T) = \operatorname{Alt}(T \otimes S) = 0.$$

(2) If $\omega \in \mathscr{T}^k(V^*)$, $\eta \in \mathscr{T}^\ell(V^*)$ and $\theta \in \mathscr{T}^m(V^*)$, then

$$\operatorname{Alt}\left(\operatorname{Alt}(\omega \otimes \eta) \otimes \theta\right) = \operatorname{Alt}\left(\omega \otimes \eta \otimes \theta\right) = \operatorname{Alt}\left(\omega \otimes \operatorname{Alt}(\eta \otimes \theta)\right).$$

(3) If $\omega \in \Lambda^k(V^*)$, $\eta \in \Lambda^\ell(V^*)$ and $\theta \in \Lambda^m(V^*)$, then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+\ell+m)!}{k!\,\ell!\,m!} \operatorname{Alt}\left(\omega \otimes \eta \otimes \theta\right).$$

Proof

(1) For $v_1, \ldots, v_{k+\ell} \in V$, since

$$(k+\ell)!\operatorname{Alt}(S \otimes T)(v_1, \ldots, v_{k+\ell}) = \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}\sigma\, S(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

Let $G = \{\sigma \in S_{k+\ell} \mid \sigma(j) = j \text{ if } k+1 \leq j \leq k+\ell\}$, i.e. G consists of all σ which leaves $k+1, \ldots, k+\ell$ fixed. Then $G \subset S_{k+\ell}$ is a subgroup of $S_{k+\ell}$, $G \cong S_k$ (i.e. there is an isomorphism mapping each $\sigma \in G$ to $\sigma' \in S_k$) and

$$\begin{aligned}
& \sum_{\sigma \in G} \operatorname{sgn}\sigma\, S(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\
= & \sum_{\sigma \in G} \operatorname{sgn}\sigma\, S(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot T(v_{k+1}, \ldots, v_{k+\ell}) \\
= & \left[\sum_{\sigma' \in S_k} \operatorname{sgn}\sigma'\, S(v_{\sigma'(1)}, \ldots, v_{\sigma'(k)})\right] \cdot T(v_{k+1}, \ldots, v_{k+\ell}) \\
= & \; k!\operatorname{Alt}(S)(v_1, \ldots, v_k) \cdot T(v_{k+1}, \ldots, v_{k+\ell}) = 0
\end{aligned}$$

Suppose $\sigma_0 \notin G$. Let $G \cdot \sigma_0 = \{\sigma' \cdot \sigma_0 \mid \sigma' \in G\}$, i.e. a right coset of G in $S_{k+\ell}$, and let

$$v_{\sigma_0(1)} = w_1, \ldots, v_{\sigma_0(k+\ell)} = w_{k+\ell}$$

Then

$$\begin{aligned}
& \sum_{\sigma = \sigma' \cdot \sigma_0 \in G \cdot \sigma_0} \operatorname{sgn}\sigma\, S(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\
= & \operatorname{sgn}\sigma_0 \sum_{\sigma' \in G} \operatorname{sgn}\sigma'\, S(w_{\sigma'(1)}, \ldots, w_{\sigma'(k)}) \cdot T(w_{\sigma'(k+1)}, \ldots, w_{\sigma'(k+\ell)}) \\
= & \left[\operatorname{sgn}\sigma_0 \sum_{\sigma' \in G} \operatorname{sgn}\sigma'\, S(w_{\sigma'(1)}, \ldots, w_{\sigma'(k)})\right] \cdot T(w_{k+1}, \ldots, w_{k+\ell}) \\
= & \; (-1)\; k!\operatorname{Alt}(S)(v_1, \ldots, v_k) \cdot T(v_{k+1}, \ldots, v_{k+\ell}) = 0
\end{aligned}$$

Notice that $G \cap G \cdot \sigma_0 = \emptyset$. In fact, if $\sigma \in G \cap G \cdot \sigma_0$, then there exists a $\sigma' \in G$ such that $\sigma = \sigma' \cdot \sigma_0 \implies \sigma_0 = \sigma \cdot (\sigma')^{-1} \in G$, a contradiction. We can continue in this way, breaking $S_{k+\ell}$

up into disjoint subsets (i.e. right cosets) such that the sum over each right coset of G in $S_{k+\ell}$ is 0, so that the sum over $S_{k+\ell}$ is 0. The relation $\text{Alt}\,(T \otimes S) = 0$ is proved similarly.

(2) and (3) Since

$$\begin{aligned}
& \text{Alt}\ (\text{Alt}\,(\eta\otimes\theta) - \eta\otimes\theta) = \text{Alt}\,(\eta\otimes\theta) - \text{Alt}\,(\eta\otimes\theta) = 0 \\
\overset{(1)}{\Longrightarrow}\ & 0 = \text{Alt}\ (\omega\otimes[\text{Alt}\,(\eta\otimes\theta) - \eta\otimes\theta]) = \text{Alt}\ (\omega\otimes\text{Alt}\,(\eta\otimes\theta)) - \text{Alt}\ (\omega\otimes\eta\otimes\theta) \\
\Longrightarrow\ & \text{Alt}\ (\omega\otimes\text{Alt}\,(\eta\otimes\theta)) = \text{Alt}\ (\omega\otimes\eta\otimes\theta) \\
\Longrightarrow\ & \omega\wedge(\eta\wedge\theta) = \frac{(k+\ell+m)!}{k!\,(\ell+m)!}\,\frac{(\ell+m)!}{\ell!\,m!}\,\text{Alt}\ (\omega\otimes\text{Alt}\,(\eta\otimes\theta)) = \frac{(k+\ell+m)!}{k!\,\ell!\,m!}\,\text{Alt}\ (\omega\otimes\eta\otimes\theta)
\end{aligned}$$

and similarly since

$$\begin{aligned}
& \text{Alt}\ (\text{Alt}\,(\omega\otimes\eta) - \omega\otimes\eta) = \text{Alt}\,(\omega\otimes\eta) - \text{Alt}\,(\omega\otimes\eta) = 0 \\
\overset{(1)}{\Longrightarrow}\ & 0 = \text{Alt}\ ([\text{Alt}\,(\omega\otimes\eta) - \omega\otimes\eta]\otimes\theta) = \text{Alt}\ (\text{Alt}\,(\omega\otimes\eta)\otimes\theta) - \text{Alt}\ (\omega\otimes\eta\otimes\theta) \\
\Longrightarrow\ & \text{Alt}\ (\text{Alt}\,(\omega\otimes\eta)\otimes\theta) = \text{Alt}\ (\omega\otimes\eta\otimes\theta) \\
\Longrightarrow\ & (\omega\wedge\eta)\wedge\theta = \frac{(k+\ell+m)!}{(k+\ell)!\,m!}\,\frac{(k+\ell)!}{k!\,\ell!}\,\text{Alt}\ (\text{Alt}\,(\omega\otimes\eta)\otimes\theta) = \frac{(k+\ell+m)!}{k!\,\ell!\,m!}\,\text{Alt}\ (\omega\otimes\eta\otimes\theta)
\end{aligned}$$

Exercise

8. Let $\varphi \in V^* \cong \Lambda^1(V^*)$. Show that $\varphi\wedge\varphi = 0$.
9. Let $\tau \in \Lambda^k(V^*)$ and $\mu \in \Lambda^\ell(V^*)$. Show that $\mu\wedge\tau = (-1)^{k\ell}\tau\wedge\mu$.

Remark Naturally $\omega\wedge(\eta\wedge\theta)$ and $(\omega\wedge\eta)\wedge\theta$ are both denoted simply $\omega\wedge\eta\wedge\theta$, and higher-order products $\omega_1\wedge\cdots\wedge\omega_r$ are defined similarly. If $v_1,\ldots,v_n$ is a basis for V and $\varphi_1,\ldots,\varphi_n$ is the dual basis, a basis for $\Lambda^k(V^*)$ can be constructed quite easily.

Theorem For $k \le n$, the set

$$\{\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_n \le n\}$$

is a basis for $\Lambda^k(V^*)$, which therefore has dimension

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Proof If $\omega \in \Lambda^k(V^*) \subset \mathscr{T}^k(V^*)$, then we can write

$$\begin{aligned}
& \omega = \sum_{1\le i_1,\ldots,i_k\le n} a_{i_1,\ldots,i_k}\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k} \\
\Longrightarrow\ & \omega = \text{Alt}\ (\omega) = \sum_{1\le i_1,\ldots,i_k\le n} a_{i_1,\ldots,i_k}\text{Alt}\ (\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}) \\
\Longrightarrow\ & \omega = \frac{1}{k!}\sum_{1\le i_1,\ldots,i_k\le n} a_{i_1,\ldots,i_k}\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k} \\
\Longrightarrow\ & \omega = \sum_{1\le i_1<\cdots<i_k\le n} b_{i_1,\ldots,i_k}\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}, \quad \text{where } b_{i_1\ldots i_k} = \frac{1}{k!}\sum_{\sigma\in S_k}(-1)^\sigma a_{i_{\sigma(1)}\cdots i_{\sigma(k)}},
\end{aligned}$$

and we have used the fact that $\varphi_i\wedge\varphi_i = 0$ for all $1 \le i \le n$.

If $\sum_{1\le i_1<\cdots<i_k\le n} a_{i_1\ldots i_k}\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}=0$, then for any $1\le j_1<\cdots<j_k\le n$,

$$\begin{aligned}
&\sum_{1\le i_1<\cdots<i_k\le n} a_{i_1\ldots i_k}\left(\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}\right)\left(v_{j_1},\ldots,v_{j_k}\right)=0\\
\Longrightarrow\quad & k!\sum_{1\le i_1<\cdots<i_k\le n} a_{i_1\ldots i_k}\mathrm{Alt}\left(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}\right)\left(v_{j_1},\ldots,v_{j_k}\right)=0
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow\quad & 0=\sum_{1\le i_1<\cdots<i_k\le n} a_{i_1\ldots i_k}\sum_{\sigma\in S_k}(-1)^\sigma\varphi_{i_1}\left(v_{j_{\sigma(1)}}\right)\cdots\varphi_{i_k}\left(v_{j_{\sigma(k)}}\right)\\
\Longrightarrow\quad & 0=\sum_{1\le i_1<\cdots<i_k\le n} a_{i_1\ldots i_k}\sum_{\sigma\in S_k}(-1)^\sigma\delta_{i_1 j_{\sigma(1)}}\cdots\delta_{i_k j_{\sigma(k)}}=a_{j_1\ldots j_k}\\
\Longrightarrow\quad & a_{j_1\ldots j_k}=0\quad\text{for all } 1\le j_1<\cdots<j_k\le n
\end{aligned}$$

Thus, $\{\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k}\mid 1\le i_1,\ldots,i_k\le n\}$ are linearly independent and form a basis for $\Lambda^k(V^*)$.

Remark If V is an n-dimensional vector space over $\mathbb{R}$, then $\dim\Lambda^n(V^*)=1$. Thus all alternating n-tensors on V are multiples of any non-zero one. Since the determinant is an example of such a member of $\Lambda^n(\mathbb{R}^n)$, it is not surprising to find it in the following theorem.

Theorem Let $\{v_i\}_{i=1}^n$ be a basis for V, and let $\omega\in\Lambda^n(V^*)$. If $w_i=\sum_{j=1}^n a_{ij}v_j$ are n vectors in V, then

$$\omega(w_1,\ldots,w_n)=\det(a_{ij})\cdot\omega(v_1,\ldots,v_n)$$

Proof Define $\eta\in\mathscr{T}^n(\mathbb{R}^n)$ by

$$\eta\left((a_{11},\ldots,a_{1n}),\ldots,(a_{n1},\ldots,a_{nn})\right)=\omega\left(\sum_{j=1}^n a_{1j}v_j,\ldots,\sum_{j=1}^n a_{nj}v_j\right)$$

Clearly $\eta\in\Lambda^n(\mathbb{R}^n)$, so there is a $\lambda\in\mathbb{R}$ such that

$$\eta=\lambda\cdot\det\implies\lambda=\eta(e_1,\ldots,e_n)=\omega(v_1,\ldots,v_n),\ \text{where } \{e_i\}_{i=1}^n \text{ is the standard basis for } \mathbb{R}^n.$$

Remark

(1) If we set $\mathscr{G}(V^*)=\sum_{k=0}^n\oplus\Lambda^k(V^*)$, where $\Lambda^0(V^*)=\mathbb{R}^1$, then $\mathscr{G}(V^*)$ is generated by 1 and $\Lambda^1(V^*)\cong V^*$ with $\dim\mathscr{G}(V^*)=2^n$. Also note that the wedge product $\wedge$ can be extended to $\mathscr{G}(V^*)$ by linearity, that is, by requiring that $\wedge$ be distributive with respect to vector addition. This multiplication $\wedge$ is associative and $\mathscr{G}(V^*)$ is an algebra, with unit 1.

(2) If $L:V^*\to V^*$ is a linear transformation, then L induces a unique algebra homomorphism $\tilde{L}:\mathscr{G}(V^*)\to\mathscr{G}(V^*)$ which extends the map L. $\tilde{L}$ preserves degrees; that is, $\tilde{L}:\Lambda^k(V^*)\to\Lambda^k(V^*)$. In particular, $\tilde{L}:\Lambda^n(V^*)\to\Lambda^n(V^*)$. Hence, since $\dim\Lambda^n(V^*)=1$, there exists a scalar λ such that $\tilde{L}\Big|_{\Lambda^n(V^*)}=\lambda i_{\Lambda^n(V^*)}$. This scalar λ is precisely *the determinant of L.*

(3) The algebra $\mathscr{G}(V^*)$ is called the *Grassmann algebra*, or *exterior algebra*, of V^*. Elements of $\mathscr{G}(V^*)$ are called *forms* on V. Forms in $\Lambda^k(V^*)$ are said to be of *degree* k.
Now let X be a smooth manifold,

$$\Lambda^k(X) = \bigcup_{x\in X} \Lambda^k(T^*(X,x)), \quad \text{and} \quad \mathscr{G}(X) = \bigcup_{x\in X} \mathscr{G}(T^*(X,x)).$$

As usual, we shall denote the projection maps from these spaces on to X by π. These spaces can each be given the structure of a smooth manifold such that π is a smooth map.

Definition A *k-form* on X is a mapping $\mu : X \to \Lambda^k(X)$ such that $\pi \circ \mu = i_X$. A k-form μ on X is *smooth* if whenever $V_1, \dots, V_k$ are *smooth vector fields* on X, then

$$\mu(V_1, \dots, V_k) \in C^\infty(X, \mathbb{R}^1), \quad \text{where } \mu(V_1, \dots, V_k)(x) = \mu(x)(V_1(x), \dots, V_k(x)).$$

A *differential form* on X is a mapping $\omega : X \to \mathscr{G}(X)$ such that $\pi \circ \omega = i_X$; it is *smooth* if its component in $\Lambda^k(X)$ is smooth for each k. The set of smooth k-forms on X is denoted by $C^\infty(X, \Lambda^k(X))$. The set of all smooth differential forms is denoted by $C^\infty(X, \mathscr{G}(X))$. Note that $C^\infty(X, \Lambda^k(X))$ is a vector space under pointwise addition and scalar multiplication, and that $C^\infty(X, \mathscr{G}(X))$ is an algebra under the additional operation of pointwise exterior multiplication.

Remark 1. A 0-form on X is just a real-valued function on X; it is a smooth 0-form if and only if it is a smooth function.

Remark 2. Let φ be a local coordinate system on X, with domain U and coordinate functions $(x_1, \dots, x_n)$. Then $\{dx_1, \dots, dx_n\}$ is a basis for $T^*(X,x)$ for each $x \in U$. Hence

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid i_1 < \cdots < i_k\} \text{ is a basis for } \Lambda^k(T^*(X,x)) \text{ for each } x \in U.$$

Thus, the restriction to U of each k-form μ on X can be expressed as

$$\mu = \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where each $a_{i_1\cdots i_k}$ is a real-valued function on U. Furthermore, μ is smooth if and only if, for each (φ, U), $a_{i_1\cdots i_k} \in C^\infty(U, \mathbb{R}^1)$. This is because

$$a_{i_1\cdots i_k} = \mu\left(\frac{\partial}{\partial x_{i_1}}, \cdots, \frac{\partial}{\partial x_{i_k}}\right).$$

Theorem 1. Let X be a smooth manifold. There exists a unique linear map $d : C^\infty(X, \mathscr{G}(X)) \to C^\infty(X, \mathscr{G}(X))$, called the *exterior differential,* such that the following properties hold.

(i) $d : C^\infty(X, \Lambda^k(X)) \to C^\infty(X, \Lambda^{k+1}(X))$;

(ii) $d(f) = df$ (ordinary differential) for $f \in C^\infty(X, \Lambda^0(X))$;

(iii) if $\mu \in C^\infty(X, \Lambda^k(X))$ and $\tau \in C^\infty(X, \mathscr{G}(X))$, then $d(\mu \wedge \tau) = (d\mu) \wedge \tau + (-1)^k \mu \wedge d\tau$; and

(iv) $d^2 = 0$.

Remark For the proof we need the following lemma, which asserts that for any exterior differentiation operator d, $(d\omega)(x)$ depends only on the behavior of ω in a small neighborhood of x.

Lemma Let $d : C^\infty(X, \mathscr{G}(X)) \to C^\infty(X, \mathscr{G}(X))$ be linear and satisfy the conditions of the theorem. Suppose $\omega \in C^\infty(X, \mathscr{G}(X))$ is such that $\omega|_W = 0$ for some open set $W \subset X$. Then

$(d\omega)|_W = 0$. Hence, if ω, $\tau \in C^\infty(X, \mathscr{G}(X))$ are such that $\omega|_W = \tau|_W$ for some open set W, then $(d\omega)|_W = (d\tau)|_W$.

Proof Suppose $\omega|_W = 0$. Let $x_0 \in W$. Let $f \in C^\infty(X, \mathbb{R}^1)$ be such that

$$f(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \notin W \end{cases} \implies (f\omega)(x) = 0 \quad \text{for all } x \in X$$

Since d is linear satisfying (ii), (iii) of the theorem,

$$0 = d(f\omega) = (df)\wedge\omega + f\,d\omega \implies d(\omega)(x_0) = 0 \implies (d\omega)|_W = 0 \quad \text{since } x_0 \text{ is an arbitrary point in } W.$$

If $\omega|_W = \tau|_W$, then $(\omega - \tau)|_W = 0$, so that

$$[d(\omega - \tau)]|_W = [d\omega - d\tau]|_W \implies d\omega|_W = d\tau|_W.$$

Proof of Theorem 1.

Uniqueness. Suppose $d : C^\infty(X, \mathscr{G}(X)) \to C^\infty(X, \mathscr{G}(X))$ satisfies the conditions of the theorem. Let $x \in X$, and let φ be a local coordinate system about x with domain U and coordinate functions $x_1, \ldots, x_n$). Let $\omega \in C^\infty(X, \Lambda^k(X))$. Then the restriction of ω to U can be expressed as

$$\omega|_U = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad \text{for some } a_{i_1 \cdots i_k} \in C^\infty(U, \mathbb{R}^1).$$

Now the right-hand side of this equation is not a differential form on X, so we cannot apply d to it. However, let U_1 be an open ball containing x with $\bar{U}_1$ compact and $\bar{U}_1 \subset U$, and let $g \in C^\infty(X, \mathbb{R}^1)$ be such that

$$g(x) = \begin{cases} 1 & \text{for } x \in U_1, \\ 0 & \text{for } x \notin U. \end{cases}$$

Then $\tilde{\omega} \in C^\infty(X, \Lambda^k(X))$, where

$$\tilde{\omega} = \sum_{i_1 < \cdots < i_k} (g a_{i_1 \cdots i_k})\, d(g x_{i_1}) \wedge \cdots \wedge d(g x_{i_k}).$$

Here, by gh, for $h \in C^\infty(U, \mathbb{R}^1)$, is meant the smooth function on X defined by

$$(gh)(x) = \begin{cases} g(x)h(x) & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

Furthermore, $\tilde{\omega}|_{U_1} = \omega|_{U_1}$. By the lemma, $(d\omega)|_{U_1} = (d\tilde{\omega})|_{U_1}$. Now

$$\begin{aligned} d\tilde{\omega} &= \sum_{i_1 < \cdots < i_k} d\left[(g a_{i_1 \cdots i_k})\, d(g x_{i_1}) \wedge \cdots \wedge d((g x_{i_k})\right] \quad \text{(by linearity)} \\ &= \sum_{i_1 < \cdots < i_k} d(g a_{i_1 \cdots i_k}) \wedge d(g x_{i_1}) \wedge \cdots \wedge d(g x_{i_k}) \\ &\quad + \sum_{i_1 < \cdots < i_k} (g a_{i_1 \cdots i_k})\, d\left[d(g x_{i_1}) \wedge \cdots \wedge d(g x_{i_k})\right] \quad \text{(by Property } (iii) \\ &= \sum_{i_1 < \cdots < i_k} d(g a_{i_1 \cdots i_k}) \wedge d(g x_{i_1}) \wedge \cdots \wedge d(g x_{i_k}), \end{aligned}$$

since each term of the second sum is zero by Properties (iii) and (iv). In particular, since g is identically 1 on U_1, and since $(d\omega)|_{U_1} = (d\tilde{\omega})|_{U_1}$,

$$(d\omega)|_{U_1} = \sum_{i_1<\cdots<i_k} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (a_{i_1\cdots i_k})\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Thus if d exists, its value at x on k-forms must be given by this formula. Since x was arbitrary in X, and since every differential form is a sum of k-forms, $k \in \{0, 1, \ldots, n\}$, uniqueness is established.

Existence. We first define d locally. Let φ be a local coordinate system on X with domain U and coordinate functions $(x_1, \ldots, x_n)$. (Note that U is itself a smooth manifold.) Define $d_U : C^\infty(U, \mathscr{G}(U)) \to C^\infty(U, \mathscr{G}(U))$ as follows. For

$$\omega = \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in C^\infty(U, \Lambda^k(U)),$$

define

$$\boxed{d_U\omega = \sum_{i_1<\cdots<i_k} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (a_{i_1\cdots i_k})\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.}$$

Extend d_U to $C^\infty(U, \mathscr{G}(U))$ by linearity. Then Properties (i) and (ii) are clearly satisfied. To verify (iii) and (iv), note first that each form in $C^\infty(U, \mathscr{G}(U))$ is a sum of forms of the type $a_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. By the linearity of d_U, together with distributivity of exterior multiplication with respect to addition, it suffices to check (iii) and (iv) on forms of this type.

Property (iii)**.** Suppose

$$\mu = a_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad \text{and} \quad \tau = b_{j_1\cdots j_\ell}\, dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}.$$

Then

$$\begin{aligned}
d_U(\mu \wedge \tau) &= d_U\left[a_{i_1\cdots i_k} b_{j_1\cdots j_\ell} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}\right] \\
&= \sum_{r=1}^{n} \left[\frac{\partial}{\partial x_r}(a_{i_1\cdots i_k})\, b_{j_1\cdots j_\ell} + a_{i_1\cdots i_k} \frac{\partial}{\partial x_r}(b_{j_1\cdots j_\ell})\right] dx_r \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&= \left(\sum_{r=1}^{n} \frac{\partial}{\partial x_r}(a_{i_1\cdots i_k})\, dx_r \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \wedge (b_{j_1\cdots j_\ell}\, dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}) \\
&\quad + (-1)^k (a_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \wedge \left(\sum_{r=1}^{n} \frac{\partial}{\partial x_r}(b_{j_1\cdots j_\ell})\, dx_r \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}\right) \\
&= (d_U\mu) \wedge \tau + (-1)^k \mu \wedge d_U\tau.
\end{aligned}$$

Property (iv)**.** For $\mu = a_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$,

$$\begin{aligned}
d_U^2\mu &= d_U\left[\sum_{r=1}^{n} \frac{\partial}{\partial x_r}(a_{i_1\cdots i_k})\, dx_r \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right] \\
&= \sum_{r,\, s=1}^{n} \frac{\partial}{\partial x_s}\left[\frac{\partial}{\partial x_r}(a_{i_1\cdots i_k})\right] dx_s \wedge dx_r \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}
\end{aligned}$$

But certainly the terms in this expression with $r = s$ are zero, since $dx_r \wedge dx_r = 0$. More over, for $r \neq s$, the equality of mixed partial derivatives on $\mathbb{R}^n$ implies that

$$\frac{\partial}{\partial x_s}\frac{\partial}{\partial x_r}\left(a_{i_1\cdots i_k}\right) = \frac{\partial}{\partial x_r}\frac{\partial}{\partial x_s}\left(a_{i_1\cdots i_k}\right),$$

so that

$$\frac{\partial}{\partial x_s}\frac{\partial}{\partial x_r}\left(a_{i_1\cdots i_k}\right) dx_s \wedge dx_r = -\frac{\partial}{\partial x_r}\frac{\partial}{\partial x_s}\left(a_{i_1\cdots i_k}\right) dx_r \wedge dx_s;$$

thus the remaining terms match up in pars which cancel each other.

Thus the operator d_U has Properties $(i)-(iv)$. By uniqueness, every linear operator on $C^\infty(U, \mathscr{G}(U))$ having these properties must be given by the above formula. In particular, if U_1 is any open subset of U, the $\varphi|_{U_1}$ is a coordinate system, and $d_{U_1} : C^\infty(U_1, \mathscr{G}(U_1)) \to C^\infty(U_1, \mathscr{G}(U_1))$ is given in the coordinate system $\varphi|_{U_1}$, by the same formula. Thus if $\omega \in C^\infty(X, \mathscr{G}(X))$, then

$$d_{U_1}\left(\omega|_{U_1}\right) = \left(d_U\left(\omega|_U\right)\right)|_{U_1}.$$

This relation enables us to define d globally by $(d\omega)|_U = d_U\left(\omega|_U\right)$ for all $\omega \in C^\infty(X, \mathscr{G}(X))$ and any coordinate neighborhoods, then

$$\left(d_U\left(\omega|_U\right)\right)|_{U\cap V} = d_{U\cap V}\left(\omega|_{U\cap V}\right) = \left(d_V\left(w|_V\right)\right)|_{U\cap V}.$$

Clearly, d has the required properties, since d_U has them for each U.

Digression on Vector Analysis

Definition A *volume element of T* is a choice of basis in $\Lambda^n(T^*)$; since $\Lambda^n(T^*)$ is 1-dimensional, a volume element is a choice of a nonzero element in $\Lambda^n(T^*)$.

Example If T is the tangent space to a manifold and $\{dx_1, \ldots, dx_n\}$ is a basis for T^*, then $dx_1 \wedge \cdots \wedge dx_n$ is a volume element of T. (Note that a volume element ω determines an isomorphism $\Lambda^n(T^*) \equiv \mathbb{R}^1$, where $r\omega$ corresponds to r. Conversely, such an isomorphism defines a volume element ω corresponding to 1.)

Remark Given a volume element ω of T, since $\Lambda^1(T^*) = T^*$, $\dim \Lambda^{n-1}(T^*) = \dim \Lambda^1(T^*)$ and T is isomorphic to its double dual T^{**}, there exists a natural isomorphism $m : \Lambda^{n-1}(T^*) \to T$ defined as follows. For $\varphi \in \Lambda^{n-1}(T^*)$, $\psi \in T^*$, $m(\varphi)$ is then defined by

$$[m(\varphi)](\psi) = \lambda, \text{ where } \lambda \text{ is the real number such that } \varphi \wedge \psi = \lambda\omega = \lambda\varphi_1 \wedge \cdots \wedge \varphi_n.$$

To show that m is an isomorphism, let $\{\varphi_1, \ldots, \varphi_n\}$ be a basis for T^* such that $\omega = \varphi_1 \wedge \cdots \wedge \varphi_n$. Then the set $\{\varphi_1 \wedge \cdots \wedge \varphi_{j-1} \wedge \varphi_{j+1} \wedge \cdots \wedge \varphi_n\}$ is a basis for $\Lambda^{n-1}(T^*)$. The value of m on these basis vectors is then given by

$$m\left(\varphi_1 \wedge \cdots \wedge \varphi_{j-1} \wedge \varphi_{j+1} \wedge \cdots \wedge \varphi_n\right) = (-1)^{n+j} e_j,$$

where $\{e_1, \ldots, e_n\}$ is the basis for T dual to $\{\varphi_1, \ldots, \varphi_n\}$.

Remark Given an inner product $\langle\, , \rangle$ on a finite dimensional vector space T, there exists a natural isomorphism $g : T \to T^*$ defined by

$$[g(v)](w) = \langle v, w\rangle \quad \text{for } v, w \in T.$$

If $\{e_1, \ldots, e_n\}$ is a basis for T, let $g_{ij} = \langle e_i, e_j\rangle$, $(1 \le i, j \le n)$. Then in terms of the dual basis $\{\varphi_1, \ldots, \varphi_n\}$ for T^*,

$$g(e_i) = \sum_{j=1}^{n} g_{ij}\varphi_j \quad \text{for } 1 \le i \le n.$$

In particular, if $\{e_1, \ldots, e_n\}$ is orthonormal, then $g_{ij} = \delta_{ij}$, and

$$g(e_i) = \varphi_i.$$

Applications Take $T = \mathbb{R}^n$. Then T has an inner product and a natural volume element $\omega = \varphi_1 \wedge \cdots \wedge \varphi_n$, where $\{\varphi\}$ is the dual basis to the natural basis $\{e_i\}$ for $\mathbb{R}^n$. Thus the isomorphism m and g are defined. Also, we have natural identifications $T(\mathbb{R}^n, x) \longleftrightarrow \mathbb{R}^n$ for each $x \in \mathbb{R}^n$.

(1) Let $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^1)$. Then the *gradient of f* is the vector field on $\mathbb{R}^n$ given by

$$\boxed{\operatorname{grad} f = g^{-1} \circ (df)\,.}$$

Relative to the usual coordinates $(x_1, \ldots, x_n) = (r_1, \ldots, r_n)$ on $\mathbb{R}^n$,

$$\operatorname{grad} f = g^{-1} \circ (df) = g^{-1} \circ \left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \right) = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_j} \longleftrightarrow \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right).$$

(2) Let V be a vector field on Take $\mathbb{R}^3$. Then $g \circ V$ is a 1-form and $d\,(g \circ V)$ is a 2-form. Now for dimension $T = 3$, $\Lambda^2(T^*) = \Lambda^{n-1}(T^*)$, so the isomorphism m maps $\Lambda^2(T^*) \to T$. Thus $m\,(d\,(g \circ V))$ is a vector field on $\mathbb{R}^3$. It is called the *curl of V.*

$$\boxed{\operatorname{curl} V = (m \circ d \circ g)\,(V).}$$

Exercise 10. Compute the coordinate expression for $\operatorname{curl} V$.

(3) Let $v_1 = (a_1, b_1, c_1)$ and $v_2 = (a_2, b_2, c_2)$ be vectors in $\mathbb{R}^3$. Then $g(v_1)$ and $g(v_2)$ are 1-forms. Their exterior product is a 2-form; its image under m is a vector, called the *cross product of v_1 and v_2.*

$$\boxed{v_1 \times v_2 = m\,(g(v_1) \wedge g(v_2))\,.}$$

(4) Let V be a vector field on $\mathbb{R}^n$. Then $m^{-1}(V)$ is an $(n-1)$-form on $\mathbb{R}^n$. Its differential is an n-form; that is, a multiple of the volume element ω. This multiple is(up to sign) the *divergence of V*:

$$\boxed{(-1)^{n-1}\; d \circ m^{-1}(V) = (\operatorname{div} V)\,\omega.}$$

Remark Using these formulas, certain important formulas of vector analysis become trivial consequences of $d^2 = 0$.

- $\operatorname{curl} \operatorname{grad} f = 0$ since

$$\operatorname{curl} \operatorname{grad} f = m \circ d \circ g\left(g^{-1} \circ (df)\right) = m\left(d^2 f\right) = 0.$$

- $\operatorname{div} \operatorname{curl} V = 0$ since

$$d \circ m^{-1}(\operatorname{curl} V) = d \circ m^{-1}(m \circ d \circ g\,(V)) = d^2\,(g(V)) = 0.$$

Definition Let X and Y be smooth manifolds, and let $\Psi : X \to Y$ be a smooth map. Then an *induced map* $\Psi^* : C^\infty(Y, \mathscr{G}(Y)) \to C^\infty(X, \mathscr{G}(X))$ is defined as follows. For $f \in C^\infty(Y, \Lambda^0(Y))$, $\Psi^*(f) = f \circ \Psi$; for $\omega \in C^\infty(Y, \Lambda^k(Y))$, $k > 0$,

$$(\Psi^*\omega)(x)(v_1, \ldots, v_k) = \omega(\Psi(x))(d\Psi(v_1), \ldots, d\Psi(v_k)) \quad \text{for } v_1, \ldots, v_k \in T(X, x),\ x \in X;$$

Ψ^* is extended to $C^\infty(Y, \mathscr{G}(Y))$ by linearity.

Remarks It is easy to check that, if ω is a smooth differential form, then so is $\Psi^*\omega$. It is clear that Ψ^* maps k-forms into k-forms. In fact, it is easily checked that Ψ^* is an algebra homomorphism; i.e. Ψ^* is linear and

$$\Psi^*(\omega \wedge \tau) = (\Psi^*\omega) \wedge (\Psi^*\tau) \quad \text{for all } \omega,\ \tau.$$

Example Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ be coordinate functions of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. Then f induces a linear transformation (called *push-forward*) $f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$, defined by

$$f_*(v_p) = (Df(p)(v))_{f(p)} \quad \text{for } v_p = (p,\ v) \in \mathbb{R}^n \times \mathbb{R}^n_p$$

This linear transformation induces a linear transformation (called *pull-back*) $f^* : \Lambda^k\left(\mathbb{R}^m_{f(p)}\right) \to \Lambda^k\left(\mathbb{R}^n_p\right)$. If ω is a k-form on $\mathbb{R}^m$ we can therefore define a k-form $f^*\omega$ on $\mathbb{R}^n$ by $(f^*\omega)(p) = f^*(\omega(f(p)))$, that is, if $v_1, \ldots, v_k \in \mathbb{R}^n_p$, then we have

$$f^*\omega(p)(v_1, \ldots, v_k) = \omega(f(p))(f_*(v_1), \ldots, f_*(v_k).)$$

Thus

(1) if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, $g : \mathbb{R}^m \to \mathbb{R}$ is a function, η, ω, ω_1 and ω_2 are differential forms on $\mathbb{R}^m$, then

- $f^*(dy_i) = \sum_{j=1}^n D_j f_i \cdot dx_j = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \cdot dx_j$ for each $1 \le i \le m$.

  Proof For each $v_p = (p, v) \in \mathbb{R}^n \times \mathbb{R}^n_p$, $1 \le i \le m$,

$$\begin{aligned}
& f^*(dy_i)(p)(v_p) \\
= \;& dy_i(f(p))(f_* v_p) = dy_i(f(p))\left(\sum_{j=1}^n v_j \cdot D_j f_1(p), \ldots, \sum_{j=1}^n v_j \cdot D_j f_m(p)\right)_{f(p)} \\
= \;& dy_i(f(p))\left(\sum_{k=1}^m \sum_{j=1}^n v_j \cdot D_j f_k(p) \frac{\partial}{\partial y_k}\right)_{f(p)} = \sum_{j=1}^n v_j \cdot D_j f_i(p) = \sum_{j=1}^n D_j f_i(p) \cdot dx_j(p)(v_p)
\end{aligned}$$

- $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$.
- $f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega$.
- $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

(2) if $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and $h : \mathbb{R}^n \to \mathbb{R}$ is a function, then

$$\begin{aligned}
& f^*(h\, dx_1 \wedge \cdots \wedge dx_n) = f^*(h) \cdot (f^* dx_1) \wedge \cdots \wedge (f^* dx_n) \\
= \;& (h \circ f) \cdot \left(\sum_{j=1}^n D_j f_1 \cdot dx_j\right) \wedge \cdots \wedge \left(\sum_{j=1}^n D_j f_n \cdot dx_j\right) = (h \circ f) \cdot \det(D_j f_i) \cdot dx_1 \wedge \cdots \wedge dx_n.
\end{aligned}$$

where $\det(D_j f_i)$ is the determinant of the $n \times n$ matrix $(\partial f_i / \partial x_j)_{1 \le i, j \le n}$.

Theorem 2. Let X and Y be smooth manifolds, and let $\Psi : X \to Y$ be a smooth map. Then

$$d \circ \Psi^* = \Psi^* \circ d.$$

Proof (1) If $f \in C^\infty(Y, \Lambda^0(Y))$, then for $v \in T(X, x)$,

$$\begin{aligned} [d \circ \Psi^*(f)](v) &= [d(f \circ \Psi)](v) \\ &= [df \circ d\Psi](v) \quad \text{(since } d \text{ on functions is ordinary differential)} \\ &= [\Psi^*(df)](v) \\ &= [(\Psi^* \circ d)(f)](v) \end{aligned}$$

(2) For ω a 1-form on Y of the type $\omega = df$,

$$\begin{aligned} (d \circ \Psi^*)(\omega) &= d(\Psi^*(df)) \\ &= d(\Psi^* \circ d(f)) \\ &= d(d \circ \Psi^*(f)) \quad \text{(by (1))} \\ &= 0, \end{aligned}$$

and

$$(\Psi^* \circ d)(\omega) = \Psi^*(d\omega) = \Psi^*(ddf) = \Psi^*(0) = 0.$$

(3) Using (1) and (2), together with the fact that Ψ^* is an algebra homomorphism, the result is established in general by checking it locally on k-forms ω restricted to local coordinate neighborhoods:

$$\omega|_U = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Definitions Let X be a smooth manifold. A smooth differential form ω on X is *closed* if $d\omega = 0$. A form ω is *exact* if it is the differential of another form on X; that is, ω is exact if $\omega = d\tau$ for some smooth form τ. (Note that every exact form is closed, since $d^2 = 0$. The converse question is fundamental to our subject.) Let

- $Z^k(X, d) = \{\omega \in \Lambda^k(X) \mid d\omega = 0\}$ be the vector space of closed k-form on X,
- $B^k(X, d) = \{\omega \in \Lambda^k(X) \mid \omega = d\tau, \text{ for some } \tau \in \Lambda^{k-1}(X)\}$ be the vector space of exact k-form on X, and note that $B^k(X, d) \subset Z^k(X, d)$ because $d^2 = 0$,
- $H^k(X, d)$ be *the k^{th} De Rham cohomology group of X* defined by

  $$H^k(X, d) = Z^k(X, d)/B^k(X, d) = \{[\omega] \mid \omega \in Z^k(X, d)\},$$

  where $[\omega] \subset Z^k(X, d)$ is the equivalence class of ω and a closed k-form $\omega_1 \in [\omega] \iff \omega_1 - \omega \in B^k(X, d)$, i.e. $\omega_1 - \omega$ is an exact k-form. Its dimension, which we shall see is finite for compact X, is called *the k^{th} Betti number of X.*

Remark Although these cohomology groups are defined in terms of the manifold structure of X, they are topological invariants; that is, if two manifolds are homeomorphic (by a necessarily smooth homeomorphism), then they have isomorphic cohomology groups. In fact, these groups can be defined directly using only the topological structure of X.

Example 1. $H^0(X, d) \equiv \mathbb{R}^1$ if X is connected. For since there are no forms of degree less than 0, $B^0(X, d) = 0$. Thus

$$H^0(X, d) = Z^0(X, d) = \{f \in C^\infty(X, \mathbb{R}^1) \mid df = 0\}.$$

If U is any connected coordinate neighborhood of X, with coordinate functions $(x_1, \ldots, x_n)$, then $df = 0$ on U means

$$0 = df = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f)\, dx_i;$$

that is, $(\partial/\partial x_i)(f) = 0$ for all $1 \le i \le n$. But this implies that f is constant on U. Since X is connected, and since f is constant on each connected coordinate neighborhood in X, then f must be constant on X; that is, $Z^0(X, d) = \{\text{constant functions on } X\} \equiv \mathbb{R}^1$.

Example 2. $H^0(\mathbb{S}^1, d) \equiv \mathbb{R}^1$, where $\mathbb{S}^1$ is the unit circle. For since there are no nonzero k-forms on $\mathbb{S}^1$ for $k > 1$, $Z^1(\mathbb{S}^1, d) = C^\infty(\mathbb{S}^1, \Lambda^1(\mathbb{S}^1))$. Moreover,

$$B^1(\mathbb{S}^1, d) = \{df \mid f \in C^\infty(\mathbb{S}^1, \mathbb{R}^1)\}.$$

Now, if θ denotes the polar coordinate on $\mathbb{S}^1$, then $\partial/\partial\theta$ is a nonzero vector field on $\mathbb{S}^1$

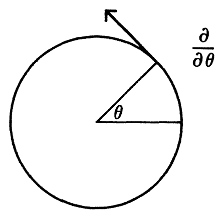


Figure 5.4

and its dual 1-form $d\theta$ is a nonzero 1-form on $\mathbb{S}^1$ (see Figure 5.4). Furthermore, $d\theta$ is not exact (since θ is not a periodic), but, given any 1-form $\omega = g(\theta)d\theta$ on $\mathbb{S}^1$, $\omega - (c\, d\theta)$ is exact for some $c \in \mathbb{R}^1$, i.e. $\omega \in [c\, d\theta] \iff \omega - (c\, d\theta) = df \in B^1(\mathbb{S}^1, d) \iff g(\theta) - c = \partial f/\partial\theta$ for some periodic function f on $\mathbb{S}^1$. Thus

$$Z^1(\mathbb{S}^1, d)/B^1(\mathbb{S}^1, d) \equiv \{c\, d\theta \mid c \in \mathbb{R}^1\} \equiv \mathbb{R}^1.$$

Exercise 11. Verify the above facts by taking $c = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} g(\theta)\, d\theta$.

Remarks Let $\psi : X \to Y$ be smooth. Note that

- if ω is a closed k-form on Y, since $d(\psi^*\omega) = \psi^*(d\omega) = \psi^*(0) = 0$, $\psi^*\omega$ is a closed k-form on X,
- if $\omega = d\tau$ is an exact k-form on Y, then $\psi^*(\omega) = \psi^*(d\tau) = d(\psi^*(\tau))$, $\psi^*\omega$ is an exact k-form on X,

This implies that

$$\psi^* : Z^k(Y, d) \to Z^k(X, d), \quad \psi^* : B^k(Y, d) \to B^k(X, d)$$

and ψ^* induces a linear map $\tilde{\psi}$ on cohomology, such that

$$\tilde{\psi} : Z^k(Y, d)/B^k(Y, d) \to Z^k(X, d)/B^k(X, d); \quad \text{that is,} \quad \tilde{\psi} : H^k(Y, d) \to H^k(X, d).$$

If $S : W \to X$ and $T : X \to Y$ are smooth, it is easy to check that $(T \circ S)^* = S^* \circ T^*$, and hence $\widetilde{(T \circ S)} = \tilde{S} \circ \tilde{T}$:

$$\begin{array}{lclcl} W & \xrightarrow{S} & X & \xrightarrow{T} & Y, \\ Z^k(W,d),\ B^k(W,d) & \xleftarrow{S^*} & Z^k(X,d),\ B^k(X,d) & \xleftarrow{T^*} & Z^k(Y,d),\ B^k(Y,d), \\ H^k(W,d) & \xleftarrow{\tilde{S}} & H^k(X,d) & \xleftarrow{\tilde{T}} & H^k(Y,d). \end{array}$$

Thus we have attached to each smooth manifold X new algebraic invariants $H^k(X,d)$ such that given smooth maps between manifolds, there are induced algebraic maps between these algebraic objects. As in the case of the fundamental group, we are thus able to solve certain difficult topological problems by studying their algebraic counterparts.

Now let us show that $H^k(\mathbb{R}^n, d) = 0$ for all $k > 0$. Since $\mathbb{R}^n$ is diffeomorphic (isomorphic as a smooth manifold) with the unit ball $B_1(0) = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ about 0 in $\mathbb{R}^n$, we may as well show that $H^k(B_1(0), d) = 0$ for all $k > 0$. For this we need the following technical lemma.

Lemma Let X be a smooth manifold. Then, for each k, consider the maps

$$C^\infty(X, \Lambda^{k-1}(X)) \underset{h_{k-1}}{\overset{d}{\rightleftarrows}} C^\infty(X, \Lambda^k(X)) \underset{h_k}{\overset{d}{\rightleftarrows}} C^\infty(X, \Lambda^{k+1}(X))$$

Suppose there exist linear maps

$$h_j : C^\infty(X, \Lambda^{j+1}(X)) \to C^\infty(X, \Lambda^j(X)) \quad (j = k-1 \text{ or } k)$$

such that $h_k \circ d + d \circ h_{k-1}$ is the identity map on $C^\infty(X, \Lambda^k(X))$. Then $H^k(X,d) = 0$; that is every closed k-form is exact.

Proof For $k \geq 1$, suppose $\omega \in C^\infty(X, \Lambda^k(X))$ is closed. Then

$$\omega = (h_k \circ d + d \circ h_{k-1})(\omega) = h_k(d\omega) + d(h_{k-1}\omega) = d(h_{k-1}\omega) \in B^k(X,d) \implies H^k(X,d) = 0$$

Remark If a sequence of such linear maps h_j is defined for all $j \geq 0$, the sequence h_j is called a *homotopy operator.*

Theorem 3 (Poincaré's Lemma) Let $U = B_1(0) \subset \mathbb{R}^n$. Then $H^k(U,d) = 0$ for all $k > 0$.

Proof To construct maps h_{k-1}, h_k satisfying the conditions of the lemma, since these maps are to be linear, it suffices to define h_{k-1} on forms $\omega = g\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$; similarly for h_k. For such ω, set

$$h_{k-1}(\omega)(x) = \left(\int_0^1 t^{k-1} g(tx)\, dt\right) \mu = \sum_{j=1}^{k} (-1)^{j-1} \left(\int_0^1 t^{k-1} g(tx)\, dt\right) x_{i_j}\, dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_k}$$

where $\mu = \sum_{j=1}^{k} (-1)^{j-1} x_{i_j}\, dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_k}$, $\widehat{dx_{i_j}}$ indicates that the term dx_{i_j} is omitted and note that

$$d\mu = \sum_{j=1}^{k} (-1)^{j-1} dx_{i_j} \wedge dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_k} = \sum_{j=1}^{k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} = k\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

The map h_k is defined similarly by replacing k everywhere by $k+1$.

Now, for $\omega = g\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in C^\infty(U, \Lambda^k(U))$ and $x \in U$,

$$\begin{aligned}
(d \circ h_{k-1})(\omega)(x) &= d\left[\left(\int_0^1 t^{k-1} g(tx)\, dt\right)\mu\right] \\
&= \sum_{\ell=1}^n \frac{\partial}{\partial x_\ell}\left(\int_0^1 t^{k-1} g(tx)\, dt\right) dx_\ell \wedge \mu + \left(\int_0^1 t^{k-1} g(tx)\, dt\right) d\mu \\
&= \sum_{\ell=1}^n \left(\int_0^1 t^{k-1} \frac{\partial}{\partial x_\ell}(g(tx))\, dt\right) dx_\ell \wedge \mu + \left(\int_0^1 t^{k-1} g(tx)\, dt\right) d\mu \\
&= \sum_{\ell=1}^n \left(\int_0^1 t^k \frac{\partial g}{\partial x_\ell}(tx)\ dt\right) dx_\ell \wedge \mu + k\left(\int_0^1 t^{k-1} g(tx)\, dt\right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \\
(h_k \circ d)(\omega)(x) &= h_k\left(\sum_{\ell=1}^n \frac{\partial g}{\partial x_\ell} dx_\ell \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \\
&= \sum_{\ell=1}^n \left(\int_0^1 t^k \frac{\partial g}{\partial x_\ell}(tx)\ dt\right)[x_\ell dx_{i_1} \wedge \cdots \wedge dx_{i_k} - dx_\ell \wedge \mu]
\end{aligned}$$

Thus

$$\begin{aligned}
(d \circ h_{k-1} + h_k \circ d)(\omega)(x) &= \left[k\left(\int_0^1 t^{k-1} g(tx)\ dt\right) + \sum_{\ell=1}^n \left(\int_0^1 t^k \frac{\partial g}{\partial x_\ell}(tx)\, x_\ell\, dt\right)\right] dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \left\{\int_0^1 \left[k t^{k-1} g(tx) + t^k \frac{d}{dt}(g(tx))\right] dt\right\} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \left\{\int_0^1 \frac{d}{dt}\left[t^k g(tx)\right]\ dt\right\} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= t^k g(tx)\Big|_0^1 dx_{i_1} \wedge \cdots \wedge dx_{i_k} = g(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \omega(x)
\end{aligned}$$

for all $x \in U$. Since $d \circ h_{k-1} + h_k \circ d$ acts as identity on such ω, it acts by linearity as identity on all k-forms.

Remark 1. Given a vector space V and $v \in V$, v defines a map, called *interior multiplication,* $i(v) : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ by

$$\begin{aligned}
& [i(v)(\omega)](v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1}). \\
\Longrightarrow\quad & i(x)(\omega) = [i(\sum_{j=1}^k x_{i_j} \partial/\partial x_{i_j})(\omega)] = \sum_{j=1}^k x_{i_j}[i(\partial/\partial x_{i_j})(g\, dx_{i_1} \wedge \cdots \wedge dx_{i_k})] \\
& = \sum_{j=1}^k (-1)^{j-1}\left(\int_0^1 t^{k-1} g(tx)\, dt\right) x_{i_j}\, dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_k} = h_{k-1}(\omega)(x),
\end{aligned}$$

that is, i is a bilinear map $V \otimes \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ and the map h_{k-1} was obtained by applying $i(x)$ to ω and averaging over the line through the origin in the direction x.

Remark 2. Poincaré lemma is a special case of a more general result. Let U be a smooth manifold. Suppose there exists a smooth map $\Psi : U \times I_\varepsilon \to U$, where $I_\varepsilon = \{r \in \mathbb{R}^1 \mid -\varepsilon < r < 1 + \varepsilon\}$, such that $\Psi(u, 1) = u$ for all $u \in U$, and $\Psi(u, 0) = u_0$ for all $u \in U$; some $u_0 \in U$ (see Figure 5.5). Then $H^k(U, d) = 0$ for all $k > 0$. The map Ψ is a *smooth homotopy*. This theorem says that if U is smoothly homotopic to a point, then the cohomology of U is that of a point.

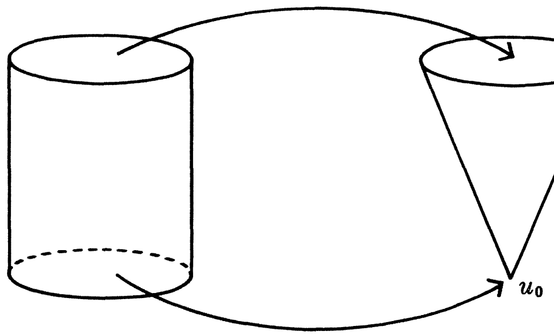
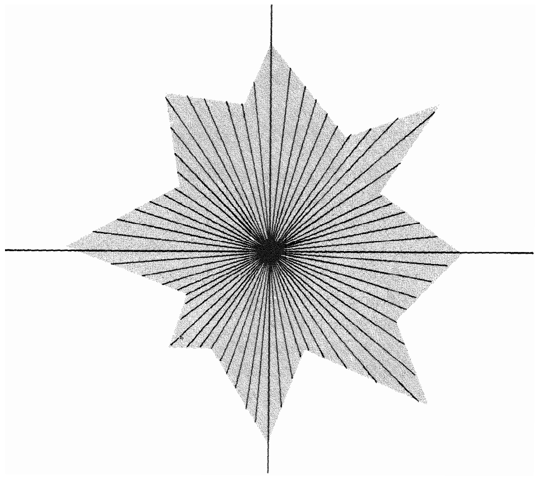


Figure 5.5



In the case covered by Poincaré lemma, a smooth homotopy is given by

$$\Psi(x,t) = tx \quad (t \in I_\varepsilon; x \in B_1(0)).$$

Note that the above proof of Poincaré's lemma works equally well for a *star-shaped region*, that is, an open set U such that for some $x_0 \in U$, the line segment joining x_0 to any other point in U lies completely in U.

§5.3 Miscellaneous Facts

Theorem 1. Let X and Y be smooth manifolds, with X connected, and let $\psi : X \to Y$ be smooth. Assume $d\psi \equiv 0$. Then ψ is a constant map; that is, $\psi(x) = y_0$ for some $y_0 \in Y$ and for all $x \in X$.

Proof Let $y_0 \in \psi(X)$. Then $\psi^{-1}(y_0)$ is a closed set in X. We shall show this set is also open, hence $\psi^{-1}(y_0) = X$ since X is connected.

Suppose $x_0 \in \psi^{-1}(y_0)$. It is sufficient to find an open set U in X such that $x_0 \in U$ and $U \subset \psi^{-1}(y_0)$. Let V be a coordinate neighborhood of y_0, with coordinate functions $(y_1, \ldots, y_m)$. Take U to be any connected coordinate neighborhood of x_0 such that $U \subset \psi^{-1}(V)$. Let $(x_1, \ldots, x_n)$ denote the coordinate functions in U. Then, for each $x \in U$, the matrix for $d\psi(x)$ relative to the bases $\{\partial/\partial x_j\}$ for $T(X,x)$ and $\{\partial/\partial y_i\}$ for $T(Y,\psi(x))$ is

$$\left(\frac{\partial}{\partial x_j}(y_i \circ \psi)\right).$$

Now, $d\psi \equiv 0$ implies $(\partial/\partial x_j)(y_i \circ \psi) \equiv 0$ on U for all i, j. But this implies that $y_i \circ \psi$ is constant on U for all i. Hence $y_i \circ \psi(x) = y_i \circ \psi(x_0)$ for all i and for all $x \in U$; that is, $\psi(x) = \psi(x_0) = y_0$ for all $x \in U$, and $U \subset \psi^{-1}(y_0)$ as required.

Definition Let X be a smooth manifold, and let V and W be smooth vector fields on X. The *bracket* $[V,\, W]$ of V and W is the smooth vector field on X defined by

$$[V,\, W](f) = V(Wf) - W(Vf) \quad \text{for } f \in C^\infty(X, \mathbb{R}^1).$$

Remark $[V,\, W]$ is a vector field, because if φ is a local coordinate system with domain U and coordinate functions $(x_1, \ldots, x_n)$, then

$$V|_U = \sum_{i=1}^n a_i \left(\frac{\partial}{\partial x_i}\right) \quad \text{and} \quad W|_U = \sum_{i=1}^n b_i \left(\frac{\partial}{\partial x_i}\right) \quad \text{for some } a_i,\, b_i \in C^\infty(U, \mathbb{R}).$$

Since $[V, W]$ is clearly bilinear, it suffices to check that $[V, W]$ is a vector field when $V = a\,(\partial/\partial x_i)$ and $W = b\,(\partial/\partial x_j)$. Then, since mixed partials are equal,

$$\begin{aligned}
[V, W](f) &= a\frac{\partial}{\partial x_i}\left(b\frac{\partial}{\partial x_j}(f)\right) - b\frac{\partial}{\partial x_j}\left(a\frac{\partial}{\partial x_i}(f)\right) \\
&= a\frac{\partial}{\partial x_i}(b)\,\frac{\partial}{\partial x_j}(f) + ab\frac{\partial}{\partial x_i}\,\frac{\partial}{\partial x_j}(f) - b\frac{\partial}{\partial x_j}(a)\,\frac{\partial}{\partial x_i}(f) - ab\frac{\partial}{\partial x_j}\,\frac{\partial}{\partial x_i}(f) \\
&= \left[a\frac{\partial}{\partial x_i}(b)\,\frac{\partial}{\partial x_j} - b\frac{\partial}{\partial x_j}(a)\,\frac{\partial}{\partial x_i}\right](f) = a\frac{\partial b}{\partial x_i}\frac{\partial f}{\partial x_j} - b\frac{\partial a}{\partial x_j}\frac{\partial f}{\partial x_i}
\end{aligned}$$

Since $a\,(\partial/\partial x_i)\,(b)$ and $b\,(\partial/\partial x_j)\,(a) \in C^\infty(X, \mathbb{R})$, $[V, W]$ is indeed a smooth vector field.

Exercise 12 Show that $C^\infty(X, T(X))$ is a *Lie algebra* under bracket multiplication, that is, the bracket of vector fields has the following properties.

(1) $[V, W] = -[W, V]$ for $V, W \in C^\infty(X, T(X))$,

(2) $[V_1 + V_2, W] = [V_1, W] + [V_2, W]$ for $V_1, V_2, W \in C^\infty(X, T(X))$,

(3) $[c\,V, W] = c\,[V, W]$ for $V, W \in C^\infty(X, T(X))$, $c \in \mathbb{R}^1$,

(4) (Jacobi identity) $[[V, W], Z] + [[W, Z], V] + [[Z, V], W] = 0$ for $V, W, Z \in C^\infty(X, T(X))$.

Note that such an algebra is non-associative.

Theorem 2. Let ω be a smooth 1-form, and let V and W be smooth vector fields on X. Then

$$d\omega\,(V, W) = V\,(\omega(W)) - W\,(\omega(V)) - \omega\,([V, W])\,.$$

Proof It suffices to verify this formula in a local coordinate neighborhood. Furthermore, since both sides are linear in ω, we need only check it on forms of the type $\omega = f\,dg$ (since every 1-form is locally a sum $\sum a_i\,dx_i$). For $\omega = f\,dg$,

$$\begin{aligned}
d\omega\,(V, W) &= (df \wedge dg)\,(V, W) \\
&= df(V)\,dg(W) - df(W)\,dg(V) \\
&= \{(Vf)\cdot(Wg) - (Wf)\cdot(Vg)\}\,.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
&\{V\,(\omega(W)) - W\,(\omega(V)) - \omega\,([V, W])\} \\
=\ &\{V\,(fdg(W)) - W\,(fdg(V)) - fdg\,([V, W])\} \\
=\ &\{V\,(f\cdot(Wg)) - W\,(f\cdot(Vg)) - f\cdot([V, W]g)\} \\
=\ &\{(Vf)\cdot(Wg) + f\cdot V(Wg) - (Wf)\cdot(Vg) - f\cdot W(Vg) - f\cdot V(Wg) + f\cdot W(Vg)\} \\
=\ &\{(Vf)\cdot(Wg) - (Wf)\cdot(Vg)\}\,.
\end{aligned}$$

Theorem 3 **(Inverse Function Theorem)** Let X and Y be smooth manifolds of dimension n. Let $\psi : X \to Y$ be a smooth map. Suppose $x_0 \in X$ is such that

$$d\psi(x_0) : T(X, x_0) \to T(Y, \psi(x_0))$$

is an isomorphism. Then there exists a neighborhood U_0 of x_0 such that

(1) $\psi|_{U_0}$ is injective,

(2) $\psi(U_0)$ is open in Y, and

(3) $\psi^{-1} : \psi(U_0) \to U_0$ is smooth.

Proof Let φ_2 be a coordinate system about $\psi(x_0)$ with domain V and coordinate functions $(y_1, \ldots, y_n)$. Let φ_1 be a coordinate system about x_0 with domain U and coordinate functions $(x_1, \ldots, x_n)$ such that

$$U = \text{domain}\, \varphi_1 \subset \psi^{-1}(V).$$

Then, relative to the bases $\{\partial/\partial x_i\}$ for $T(X, x_0)$ and $\{\partial/\partial y_i\}$ for $T(X, \psi(x_0))$, $d\psi(x_0)$ has the matrix

$$\left(\frac{\partial}{\partial x_j}\left(y_i \circ \psi\right)|_{x_0}\right), \quad \text{which is nonsingular since } d\psi(x_0) \text{ is an isomorphism.}$$

Now transfer everything to $\mathbb{R}^n$ via φ_1 and φ_2. Let $\tilde{U} = \varphi_1(U)$, $\tilde{V} = \varphi_2(V)$, and $\tilde{\psi} : \tilde{U} \to \tilde{V}$ be

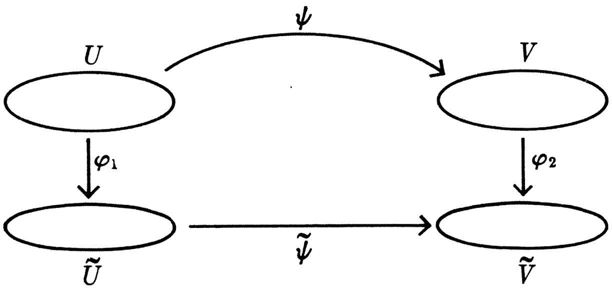


Figure 5.6

defined by $\tilde{\psi} = \varphi_2 \circ \psi \circ \varphi_1^{-1}$ (see Figure 5.6). Then $\tilde{\psi}(x) = \left(\tilde{\psi}_1(x), \ldots, \tilde{\psi}_n(x)\right)$ for $x \in \tilde{U}$, where $\tilde{\psi}_i = r_i \circ \tilde{\psi}$. The Jacobian of $\tilde{\psi}$ at $\tilde{x_0} = \varphi_1(x_0)$ is

$$\left(\left.\frac{\partial \tilde{\psi}_i}{\partial r_j}\right|_{\tilde{x_0}}\right) = \left(\frac{\partial}{\partial x_j}\left(y_i \circ \psi\right)|_{x_0}\right), \quad \text{which is nonsingular.}$$

Hence, by the classical inverse function theorem, there exists an open set $\tilde{U_0} \subset \tilde{U}$ containing $\tilde{x_0}$ such that $\tilde{V_0} = \tilde{\psi}(\tilde{U_0})$ is open, and such that the equations

$$\tilde{\psi}_i\left(r_1, \ldots, r_n\right) = s_i, \quad (1 \le i \le n)$$

have a unique solution in $\tilde{U_0}$ for each $(s_1, \ldots, s_n) \in \tilde{V_0}$. Moreover, this solution depends smoothly on $(s_1, \ldots, s_n)$. In other words, there exist smooth functions

$$h_j : \tilde{V_0} \to \mathbb{R}^1, \quad (1 \le j \le n)$$

such that for each $s = (s_1, \ldots, s_n) \in \tilde{V_0}$,

$$\tilde{\psi}_i\left(h_1(s), \ldots, h_n(s)\right) = s_i.$$

Setting $h(s) = (h_1(s), \ldots, h_n(s))$ for $s \in \tilde{V_0}$, this says that $h = \tilde{\psi}^{-1}$. Transferring back to X and Y, we find the conditions of the theorem are satisfied, with

$$U_0 = \varphi_1^{-1}\left(\tilde{U_0}\right).$$

Exercise 13 Let $A \subset \mathbb{R}^n$ be a rectangle and let $f : A \to \mathbb{R}^n$ be continuously differentiable. If there is a number M such that $|D_j f^i(x)| = |\left(\partial f^i/\partial x_j\right)(x)| \le M$ for all x in the interior of A, prove

$$|f(x) - f(y)| \le n^2 M |x - y|$$

for all x, $y \in A$, where $|f(x)-f(y)| = \left[\sum_{i=1}^{n} \left(f^i(x) - f^i(y)\right)^2\right]^{1/2}$ and $|x-y| = \left[\sum_{j=1}^{n} (x_j - y_j)^2\right]^{1/2}$.

Exercise 14 Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing a, and the differential of f is the identity matrix, i.e. $\left(D_j f^i(a)\right) = (\delta_{ij})$. Show that there is a closed rectangle U containing a in its interior such that

(a) $f(x) \ne f(a)$ if $x \in U$ and $x \ne a$.

(b) $\det\left(D_j f^i\right)(x) \ne 0$ for $x \in U$.

(c) $|D_j f^i(x) - D_j f^i(a)| < 1/2n^2$ for all i, j, and $x \in U$.

(d) $|x - y| \le 2|f(x) - f(y)|$ for x, $y \in U$.

(e) $|y - f(a)| < |y - f(x)|$ for all $y \in B_{d/2}(f(a))$, $x \in \partial U$, where $d = \min_{x\in\partial U} |f(a) - f(x)|$.

(f) for each $y \in B_{d/2}(f(a))$, there is a unique x in interior U such that $f(x) = y$.

Theorem 4 **(Implicit Function Theorem)** Let X and Y be smooth manifolds with $\dim X > \dim Y$. Let $\psi : X \to Y$ be a smooth map. Let $y_0 \in \psi(X)$ and let

$$X_0 = \psi^{-1}(y_0) = \{x \in X \mid \psi(x) = y_0\}.$$

Assume that for each $x_0 \in X$, $d\psi(x) : T(X, x) \to T(Y, \psi(x))$ is surjective. Then X_0 has a manifold structure, whose underlying topology is the relative topology of X_0 in X, and in which the inclusion map $X_0 \to X$ is smooth. Furthermore, $\dim X_0 = \dim X - \dim Y$.

Examples

(1) The n-sphere $\mathbb{S}^n$ is a smooth manifold whose topology is the induced topology in $\mathbb{R}^{n+1}$. For let $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^1$ be defined by

$$\psi(r_1, \ldots, r_{n+1}) = \sum_{i=1}^{n+1} r_i^2.$$

Then $\mathbb{S}^n = \psi^{-1}(1)$. Since $\dim \mathbb{R}^1 = 1$, we need only check that $d\psi \ne 0$ at each point of $\psi^{-1}(1)$. But $d\psi = 2\sum_{i=1}^{n+1} r_i\, dr_i$. Since $\{dr_1, \ldots, dr_{n+1}\}$ is linearly independent, $d\psi \ne 0$ unless $r_i = 0$ for all i. In particular, $d\psi \ne 0$ on $\psi^{-1}(1)$.
Note that $\dim \mathbb{S}^n = \dim \mathbb{R}^{n+1} - \dim \mathbb{R}^1 = n$, as expected.

(2) Let $X = \mathbb{R}^{n^2}$, viewed as the space of all real $n \times n$ matrices. Let $Y = \mathbb{R}^1$, an let $\psi : X \to Y$ be the determinant function. Then $\psi^{-1}(1)$ is the group of all $n \times n$ matrices of determinant 1. It is called the *unimodular group*. To verify that this group has a manifold structure, we need only show that $d\psi \ne 0$ at each point of $\psi^{-1}(1)$. Now, for (r_{ij}) coordinate functions on $\mathbb{R}^{n^2}$,

$$\psi \circ r_{ij} = \det(r_{ij}) = \sum_{\sigma\in S_n} (-1)^{\sigma} r_{1\sigma(1)} \cdots r_{n\sigma(n)}.$$

Hence
$$d\psi = \sum_{j=1}^{n} \sum_{\sigma\in S_n} (-1)^{\sigma}\, r_{1\sigma(1)} \cdots r_{j-1\sigma(j-1)} r_{j+1\sigma(j+1)} \cdots r_{n\sigma(n)} dr_{j\sigma(j)}.$$
For each $(i,\, j)$, the coefficient of dr_{ij} in this sum is, up to sign, the determinant of the cofactor of r_{ij} in (r_{ij}). These cannot all be zero at any point of $\psi^{-1}(1)$ since $\det(r_{ij}) = 1 \neq 0$ at such points. Since $\{dr_{ij} \mid 1 \le i,\, j \le n\}$ is a linearly independent set, we are done.

(3) Let $X = \mathbb{R}^{n^2}$ as in (2). Let Y be the set of all symmetric $n\times n$ real matrices. Y is a manifold, for it can be naturally identified with $\mathbb{R}^{n(n+1)/2}$: merely string out in a row the entries on and below the main diagonal. Let $\psi : X \to Y$ be defined by $\psi(x) = xx^t$ where, for each $x \in X$, x^t denotes the transpose of x. Note that ψ is smooth, since each entry of $\psi(x)$ is a polynomial in the entries of x. Let $X_0 = \psi^{-1}(1)$. Thus X_0 is the group of orthogonal $n \times n$ matrices; that is, X_0 is the *orthogonal group*.

To verify that X_0 is a manifold, we must show that $d\psi(x)$ is surjective for each $x \in X_0$. For this, it suffices to show that $d\psi(e)$ is surjective, where $e = (\delta_{ij})$ is the identity matrix. For assuming $d\psi(e)$ is surjective, let $x \in X_0$. Then the map $R_x : X \to Y$ defined by $R_x(y) = yx$ (matrix multiplication), is a smooth map with a smooth inverse, namely $R_{x^{-1}}$, and hence dR_x is everywhere an isomorphism. Moreover, $\psi \circ R_x = \psi$ for all $x \in X_0$. For if $y \in X$, then
$$\psi \circ R_x(y) = \psi(yx) = (yx)(yx)^t = yxx^ty^t = yey^t = yy^t = \psi(y).$$
Hence.
$$d\psi|_x = d\left(\psi \circ R_{x^{-1}}\right)|_x = d\psi|_{R_{x^{-1}}(x)} \circ dR_{x^{-1}}|_x,$$
so $d\psi(x)$ is a composition of surjective maps, hence is surjective.

We still must check that $d\psi(e)$ is surjective. But
$$\left(r_{ij} \circ \psi\right)(x) = \sum_{m=1}^{n} r_{im}(x) r_{jm}(x) \quad 1 \le i \le j \le n;$$
hence the entries in the matrix for $d\psi(x)$, where $1 \le k,\, \ell \le n$, and $1 \le i \le j \le n$, are
$$\frac{\partial}{\partial r_{k\ell}} (r_{ij} \circ \psi)\bigg|_x = \begin{cases} r_{j\ell}(x) & \text{if } k = i \neq j, \\ r_{i\ell}(x) & \text{if } k = j \neq i, \\ 2r_{i\ell}(x) & \text{if } k = i = j, \\ 0 & \text{otherwise.} \end{cases}$$
In particular, the entries in the matrix for $d\psi(e)$, where $1 \le k,\, \ell \le n$, $1 \le i \le j \le n$, are
$$\frac{\partial}{\partial r_{k\ell}} (r_{ij} \circ \psi)\bigg|_e = \begin{cases} r_{jj}|_e = 1 & \text{if } (k,\, \ell) = (i,\, j);\ i \neq j, \\ r_{ii}|_e = 1 & \text{if } (k,\, \ell) = (j,\, i);\ i \neq j, \\ 2r_{ii}|_e = 2 & \text{if } (k,\, \ell) = (i,\, j);\ i = j, \\ 0 & \text{otherwise.} \end{cases}$$
Thus the square submatrix, consisting of those entries with $k \le \ell$, is a diagonal matrix with diagonal entries 1 and 2, and so $d\psi(e)$ has rank $n(n+1)/2$; that is, $d\psi(e)$ is surjective.

Note that $\dim X_0 = \dim X - \dim Y = n(n-1)/2$. In fact, for any $x,\, y \in X$,
$$\left(d\psi|_x\right)(y) = \frac{d}{ds}\psi(x+sy)\bigg|_{s=0} = \frac{d}{ds}(x+sy)\left(x^t + sy^t\right)\bigg|_{s=0} = yx^t + xy^t \implies \left(d\psi|_e\right)(y) = y + y^t,$$
which implies that the Ker $(d\psi|_e) = \{y \in X \mid y + y^t = 0\}$ has dimension $n(n-1)/2$.

(4) Let X = the set of all complex $n \times n$ real matrices $= \mathbb{R}^{2n^2}$. Let $Y = \{x \in X \mid \bar{x}^t = x\}$. Let $\psi : X \to Y$ be defined by $\psi(x) = x\bar{x}^t$. Then, as in (3), the set $\psi^{-1}(e)$ is a manifold. $\psi^{-1}(e)$ is the *unitary group*. Its dimension is $2n^2 - n^2 = n^2$.

Remark Examples (2), (3), and (4) are examples of *Lie groups*; namely, they are groups whose underlying spaces are C^∞-manifolds and are such that the group operations are analytic.

Proof of the Implicit Function Theorem Let V be a coordinate neighborhood of y_0 in Y, with coordinate functions $(y_1, \dots, y_m)$. For $x_0 \in X_0$, let U be a coordinate neighborhood of x_0 in X such that $U \subset \psi^{-1}(V)$. Let $(x_1, \dots, x_n)$ denote the coordinate functions on U. We may assume that this coordinate system is chosen so that $x_i(x_0) = 0$ $(1 \le i \le n)$. Now $d\psi$ surjective at x_0 means that the $m \times n$ matrix $((\partial/\partial x_j)(y_i \circ \psi)|_{x_0})$ has rank m. By renumbering the coordinate functions on U if necessary, we may assume that the last m columns of this matrix are independent, that is, that this matrix has the form

$$(* \vdots J),$$

where J is a nonsingular $m \times m$ matrix. Let $\tilde{\psi} : U \to \mathbb{R}^{n-m} \times V$ be defined by

$$\tilde{\psi}(x) = (x_1(x), \dots, x_{n-m}(x), \psi(x)) \quad (x \in U).$$

Then $d\psi(x_0)$ has matrix

$$\begin{pmatrix} I & 0 \\ * & J \end{pmatrix},$$

where I is the identity $(n-m) \times (n-m)$ matrix. Hence $d\psi(x_0)$ is an isomorphism. By the inverse function theorem, there exists a neighborhood U_0 of x_0 such that $\tilde{\psi}|_{U_0}$ is injective, $\tilde{\psi}(U_0)$ is open in $\mathbb{R}^{n-m} \times V$, and $\tilde{\psi}^{-1} : \tilde{\psi}(U_0) \to U_0$ is smooth. We may assume that $\tilde{\psi}(U_0)$ is of the form $W_0 \times V_0$, where $0 \in W_0$ and $y_0 \in V_0$, since open sets of this type form a basis for the topology on $\mathbb{R}^{n-m} \times V$ (see Figure 5.7). Now note that $\tilde{\psi}^{-1}(W_0 \times \{y_0\}) = X_0 \cap U_0$. Since $\tilde{\psi}|_{U_0}$ is a homeomorphism, $\tilde{\psi}|_{X_0 \cap U_0}$ maps $X_0 \cap U_0$ homeomorphically onto $W_0 \times \{y_0\} \cong W_0 \subset \mathbb{R}^{n-m}$. Thus $\tilde{\psi}|_{X_0 \cap U_0}$ is a coordinate system about x_0 in X_0.

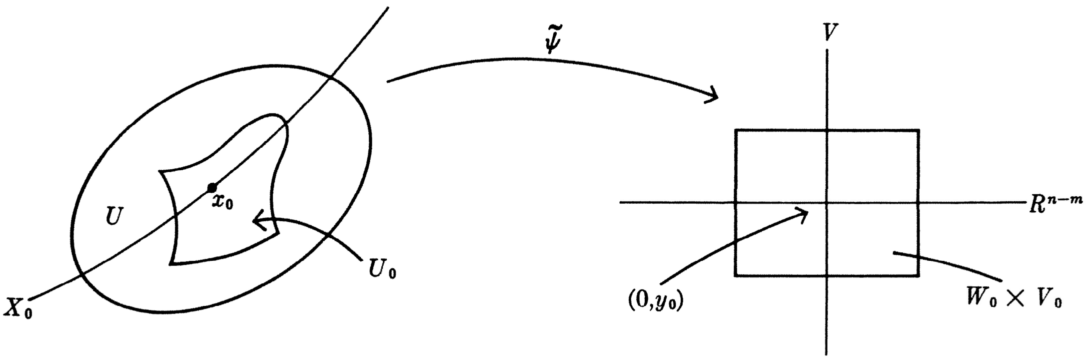


Figure 5.7

To see that such coordinate systems actually define a smooth manifold structure on X_0, we must check that they behave properly on overlaps. So suppose

$$\tilde{\psi} : U_0 \to W_0 \times V_0 \quad \text{and} \quad \tilde{\varphi} : U_1 \to W_1 \times V_1$$

are such that $(X_0 \cap U_0) \cap (X_0 \cap U_1) \neq \emptyset$ (see Figure 5.8). Since $\tilde{\psi}^{-1}$ is smooth, so is

$$\tilde{\varphi} \circ \tilde{\psi}^{-1}\Big|_{\tilde{\psi}(U_0 \cap U_1)}.$$

Restricting to $\tilde{\psi}(X_0 \cap U_0 \cap U_1) = \tilde{\psi}\,(U_0 \cap U_1) \cap \left(\mathbb{R}^{n-m} \times \{y_0\}\right)$, it follows that

$$\tilde{\varphi} \circ \tilde{\psi}^{-1}\Big|_{\tilde{\psi}(X_0 \cap U_0 \cap U_1)} : \tilde{\psi}(X_0 \cap U_0 \cap U_1) \to \tilde{\varphi}(X_0 \cap U_0 \cap U_1)$$

is smooth. Hence X_0 is a smooth manifold, of dimension $n - m$.

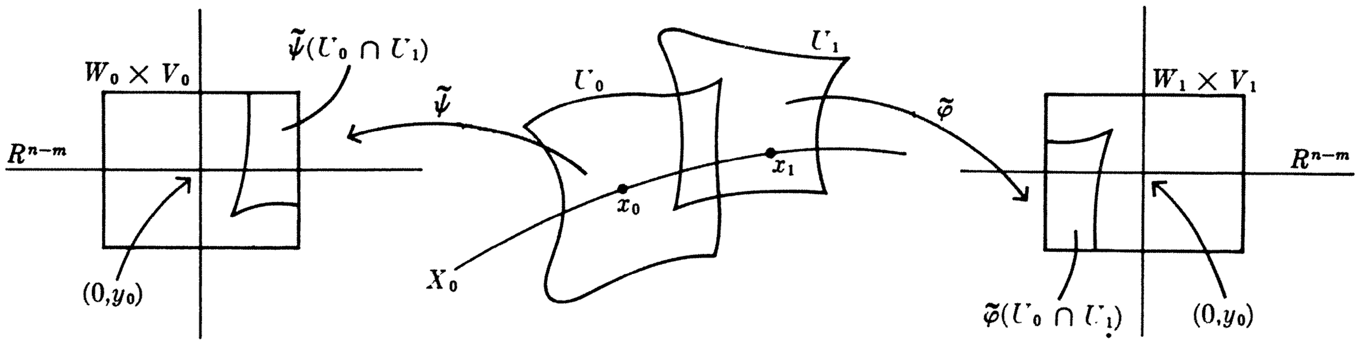


Figure 5.8

Definition A *submanifold* of a smooth manifold Y is a pair $(X,\, \psi)$, where X is a smooth manifold and $\psi : X \to Y$ is an injective smooth map such that $d\psi$ is injective at each point of X.

Examples The manifold X_0 of the previous theorem, together with the inclusion map $X_0 \to X$, is a submanifold of X. In particular, $\mathbb{S}^n$ is a submanifold of $\mathbb{R}^{n+1}$, and each of the Lie groups discussed above are submanifold of the space of all $n \times n$ real (complex in the case of the unitary group) matrices.

Remark Note that $\psi : X \to Y$ being injective does not imply that $d\psi$ is injective at each point. For example, the smooth map $\psi : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $\psi(x) = x^3$ is injective, and yet $d\psi(0) = 0$. Note also that $(X,\, \psi)$ being a submanifold of Y does not imply that ψ is a homeomorphism of X onto $\psi(X)$ with the relative topology.

Example Consider the torus

$$S^1 \times S^1 = \{(z_1,\, z_2) \mid z_1,\, z_2 \text{ are complex numbers with } |z_1| = |z_2| = 1\}.$$

Define $\psi : \mathbb{R}^1 \to S^1 \times S^1$ by $\psi(t) = \left(e^{2\pi i t},\, e^{2\pi i \alpha t}\right)$, where α is an irrational number. Then $\left(\mathbb{R}^1,\, \psi\right)$ is a submanifold of $S^1 \times S^1$. However, $\psi(\mathbb{R}^1)$ is dense in $S^1 \times S^1$, so if V is an open neighborhood of $\psi(t)$ in $S^1 \times S^1$, since $\overline{V \cap \psi(\mathbb{R}^1)} = V$, $V \cap \psi(\mathbb{R}^1) \neq \psi(U) = \left(\psi^{-1}\right)^{-1}(U)$ for any open interval $U \subset \mathbb{R}^1$ containing t, $\psi^{-1} : \psi(\mathbb{R}^1) \to \mathbb{R}^1$ is not continuous and ψ is not a homeomorphism. This submanifold is called the *skew line* on the torus. Representing the torus as a square with opposite sides identified, ψ maps $\mathbb{R}^1$ as in Figure 5.9.

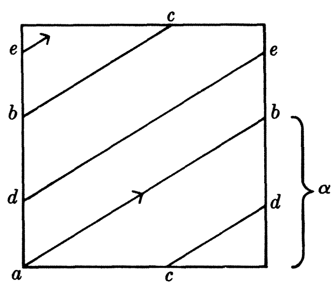


Figure 5.9

Theorem 5 Let (X, ψ) be a compact submanifold of Y. Suppose X has dimension m and Y has dimension n, where $m \leq n$. Then for each $x_0 \in X$, there exists a coordinate system $\varphi_Y : V \to \mathbb{R}^n$ about $\psi(x_0)$ with coordinate functions $(y_1, \ldots, y_n)$, such that

$$\psi(X) \cap V = \{y \in V \mid y_{m+1}(y) = \cdots = y_n(y) = 0\}.$$

Furthermore, a coordinate system $\varphi_X : U \to \mathbb{R}^m$ can be chosen about x_0 with coordinate functions $(x_1, \ldots, x_m)$, such that $U \subset \psi^{-1}(V)$ and such that $x_j = y_j \circ \psi$ for all $j \leq m$. Thus, on U,

$$y_j \circ \psi = \begin{cases} x_j & \text{for } 1 \leq j \leq m, \\ 0 & \text{for } m < j \leq n. \end{cases}$$

Proof Using the Inverse Function Theorem and the Implicit Function Theorem (in Michael Spivak Calculus of Manifolds), one can prove the following.

Proposition Let $M \subset N$, dim $M = m < n =$dim N, be a smooth submanifold and let $p \in M$ be arbitrary. Then there exists a chart $(\varphi = (x_1, \ldots, x_n),\, U)$ on N, such that $U \cap M$ neighbourhood of p in M and

$$x_{m+1}(q) = \cdots = x_n(q) = 0 \quad \text{for all } q \in U \cap M.$$

The first m entries in φ are a local coordinate system on M near p.

Proof

- fix $p \in M \subset N$, choose local coordinates $\varphi' = (x_1, \ldots, x_n)$ on N and $\psi' = (y_1, \ldots, y_m)$ on M covering p
- M being a submanifold means inclusion $i : M \to N$ is an embedding, thus di_p is in particular injective
- hence, the Jacobi matrix of i at p in local coordinates φ', ψ',

$$\left(\frac{\partial x_i}{\partial y_j}(p)\right)_{ij} \in \text{Mat}\,(n \times m,\, \mathbb{R})$$

  has maximal rank m
- w.l.o.g. (after possibly re-ordering the x_i) assume that first m rows are linearly independent
- implicit function theorem $\Longrightarrow$ $(x_1, \ldots, x_m)$ are local coordinates on some open set $V \subset M$
- after possibly shrinking V obtain that

$$q \in i(M) \iff x_k(q) = f_k(x_1(q), \ldots, x_m(q)),$$

  where $f_k : (x_1, \ldots, x_m)(V) \to \mathbb{R}$ is uniquely determined for all $m + 1 \leq k \leq n$
- choose $U \subset N$ open, such that $(x_1, \ldots, x_n)$ are defined on U and $U \cap M = V$, define for $m + 1 \leq k \leq n$

$$F_k := x_k - f_k(x_1, \ldots, x_m)$$

- define new coordinate system $(\varphi,\, U)$ on N fulfilling statement of this proposition as follows:

$$\varphi = (x_1, \ldots, x_m,\, F_{m+1}, \ldots, F_n)$$

- Jacobi matrix of φ at p with respect to the coordinates $(x_1, \ldots, x_n)$ is of the form

$$\begin{pmatrix} I_{m\times m} & 0 \\ A & I_{(n-m)\times(n-m)} \end{pmatrix} \quad \text{for some } A \in \text{Mat}\,((n-m) \times m,\, \mathbb{R}),$$

  hence invertible

- hence, φ is a local diffeomorphism and thereby defines a local coordinate system on N which, by construction, fulfils

$$\varphi(U \cap M) = \varphi(V) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

  as required

Remark When a coordinate system φ_Y is chosen as in Theorem 5, $\psi(X) \cap V$ is said to be a *slice* in ψ_Y. Note that the coordinate system obtained in the proof of Theorem 4 are of this type.

Corollary If (X, ψ) is a compact submanifold of Y, then

$$\psi : X \to \psi(X)$$

is a homeomorphism. Moreover, for each submanifold obtained by applying the implicit function theorem, the inclusion map is a homeomorphism.

Proof Since $\psi(X)$ is Hausdorff in the relative topology, the first statement is proved.

Definition Let X be a smooth manifold, and let V be a smooth vector field on X. An *integral curve* of V is a smooth curve $\alpha : (a, b) \to X$ (Figure 5.10), such that the tangent vector to α at each point is equal to the value of V at that point, that is,

$$\dot{\alpha}(t) = V(\alpha(t)) \quad \text{for all } t \in (a, b).$$

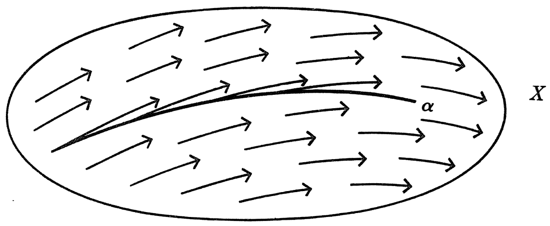


Figure 5.10

Remark Let $\varphi : U \to \mathbb{R}^n$ be a local coordinate system on X, with coordinate functions $(x_1, \ldots, x_n)$. Let $\alpha : (a, b) \to U$ be a smooth curve in U. Then, by definition, $\dot{\alpha}(t) = d\alpha(d/dt)$. Hence, the ith component of $\dot{\alpha}$ relative to the basis $\{\partial/\partial x_j\}$ is

$$dx_i(\dot{\alpha}) = dx_i\left(d\alpha\left(\frac{d}{dt}\right)\right) = d(x_i \circ \alpha)\left(\frac{d}{dt}\right) = \frac{d}{dt}(x_i \circ \alpha),$$

so that

$$\dot{\alpha} = \sum_{i=1}^{n} \frac{d}{dt}(x_i \circ \alpha) \frac{\partial}{\partial x_i}$$

Thus α is an integral curve of a vector field $V = \sum_{i=1}^{n} a_i(\partial/\partial x_i)$ if and only if

$$(*) \qquad \frac{d}{dt}(x_i \circ \alpha) = a_i \quad \text{for each } 1 \leq i \leq n.$$

Thus, to find integral curves of a given vector field V on a coordinate neighborhood U, we need to solve the system $(*)$ of differential equations. Solutions are guaranteed by the following classical theorem.

Theorem 6 Let W be an open set in $\mathbb{R}^n$, let $w_0 \in W$, and let $a_i \in C^\infty(W, \mathbb{R}^1)$, $(1 \le i \le n)$. Then there exists an open set $W_0 \subset W$ about w_0, an interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}^1$, and a smooth map $\psi : (-\varepsilon, \varepsilon) \times W_0 \to W$ such that, for each $w \in W_0$, $\psi|_{(-\varepsilon,\varepsilon)\times\{w\}}$ is a solution of the equations

$$\frac{df_i}{dt} = a_i\left(f_1(t), \ldots, f_n(t)\right) \quad (1 \le i \le n)$$

subject to the initial conditions $f_i(0) = w_i$; that is, if $\alpha_w : (-\varepsilon, \varepsilon) \to W$ is defined by

$$\alpha_w(t) = \psi(t, w),$$

then for $1 \le i \le n$,

$$\begin{array}{lll} (Eqn) & \dfrac{d}{dt}\left(r_i \circ \alpha_w\right)(t) = a_i\left(r_1 \circ \alpha_w(t), \ldots, r_n \circ \alpha_w(t)\right) & \text{for all } t \in (-\varepsilon, \varepsilon), \\ (IC) & \left(r_i \circ \alpha_w\right)(0) = r_i(w) & \text{for all } 1 \le i \le n. \end{array}$$

Furthermore, α_w is the unique function $\alpha_w : (-\varepsilon, \varepsilon) \to W$ satisfying (Eqn) and (IC).

Reinterpreting Theorem 6 in terms of vector fields, we obtain Theorem 7.

Theorem 7 Let X be a smooth manifold and let V be a smooth vector field on X. Let $x_0 \in X$. Then there exist an open set U about x_0, $(-\varepsilon, \varepsilon) \subset \mathbb{R}^1$, and a smooth map $\psi : (-\varepsilon, \varepsilon) \times U \to X$, such that for each $u \in U$, the curve

$$\alpha_u : (-\varepsilon, \varepsilon) \to X$$

defined by $\alpha_u(t) = \psi(t, u)$ is the unique integral curve from $(-\varepsilon, \varepsilon)$ into V satisfying $\alpha_u(0) = u$. Furthermore, the smooth maps $\psi_t : U \to X$, defined for each $t \in (-\varepsilon, \varepsilon)$ by $\psi_t(u) = \psi(t, u)$, have the properties

(1) $\psi_{t_1+t_2} = \psi_{t_1} \circ \psi_{t_2}$ on $\psi_{t_2}^{-1}(U)$ whenever t_1, t_2 and $t_1 + t_2 \in (-\varepsilon, \varepsilon)$,

(2) $\psi_{-t} = \psi_t^{-1}$ on $\psi_t(U) \cap U$ for each $t \in (-\varepsilon, \varepsilon)$.

Proof Let W be a coordinate system about w_0, with coordinate functions $(x_1, \ldots, x_n)$. Then, on W, $V = \displaystyle\sum_{i=1}^{n} a_i\left(\partial/\partial x_i\right)$ for some smooth functions $a_i \in C^\infty(W, \mathbb{R}^1)$.

By Theorem 6, there exist $U \subset W$, $(-\varepsilon, \varepsilon) \subset \mathbb{R}^1$, and $\psi : (-\varepsilon, \varepsilon) \times U \to W \subset X$ with the required properties. The last statement is a consequence of the uniqueness of the solution; namely, it is easy to check that

$$t_1 \to \psi(t_2 + t_1, u) \quad \text{and} \quad t_1 \to \psi(t_1, \psi_{t_2}(u))$$

are both integral curves of V which send 0 into $\psi_{t_2}(u)$, and hence they are equal; that is, $\psi_{t_1+t_2} = \psi_{t_1} \circ \psi_{t_2}$. Similarly, $\psi_{-t} = \psi_t^{-1}$.

Remark Properties (1) and (2) of Theorem 7 express the fact that ψ_t is a local *one-parameter group of transformations.*

Remark The previous theorem guarantees the existence locally of integral curves for vector fields. However, it is not always possible to obtain integral curves globally; that is, it is not possible in general to find a curve $\alpha : \mathbb{R}^1 \to X$ through x_0 such that α is an integral curve of a given vector field V. For example, let $X = \mathbb{R}^2 \setminus \{0\}$ and let $V = \partial/\partial r_1$. Then the integral curve of V through $(-1, 0)$ cannot be extended to values of $t \ge 1$. (see Figure 5.11).

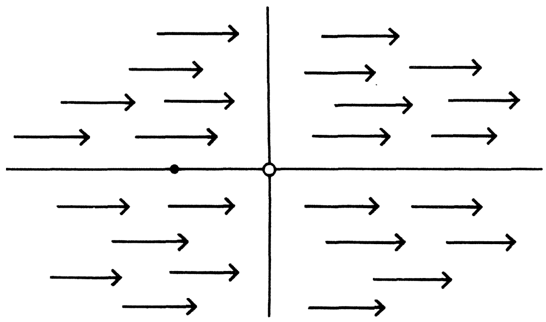


Figure 5.11

However, if X is compact, then every vector field admits through each point integral curves defined on all of $\mathbb{R}^1$.

Remark In studying the motion of a particle in $\mathbb{R}^3$ under the influence of a force field F, Newton's law tells us that the path of motion is a curve $\alpha(t) = (x_1(t), x_2(t), x_3(t))$ such that

$$m\frac{d^2x_i(t)}{dt^2} = F_i \quad (1 \leq i \leq 3),$$

where m is the mass of the particle. Setting $p_i = m(dx_i/dt)$, we have

$$\frac{dx_i}{dt} = \frac{p_i}{m}, \quad \frac{dp_i}{dt} = F_i \quad (1 \leq i \leq 3).$$

But $(x_1, x_2, x_3, p_1, p_2, p_3)$ may be regarded as coordinate functions on the cotangent bundle of $\mathbb{R}^3$. Hence the orbit of the particle is just the projection onto $\mathbb{R}^3$ of the integral curve of a vector field on the cotangent bundle. In fact, the cotangent bundle is the natural domain for the study of mechanics on a manifold.

Remark The use of integral curves provides a geometric interpretation of the bracket of two vector fields. Let V and W be smooth vector fields on X, and let $x_0 \in X$. Suppose we move along the integral curve of V through x_0 until the parameter has moved from 0 to $\sqrt{s}$; then move along an integral curve of W from 0 to $\sqrt{s}$; then move back along an integral curve of V, the parameter now varying from 0 to $-\sqrt{s}$; and finally move back along an integral curve of W from 0 to $-\sqrt{s}$; as in Figure 5.12. We will not in general return to our starting point. As $s \to 0$, our end point will trace out a curve through x_0. The bracket $[V, W](x_0)$ is precisely the tangent vector to this curve.

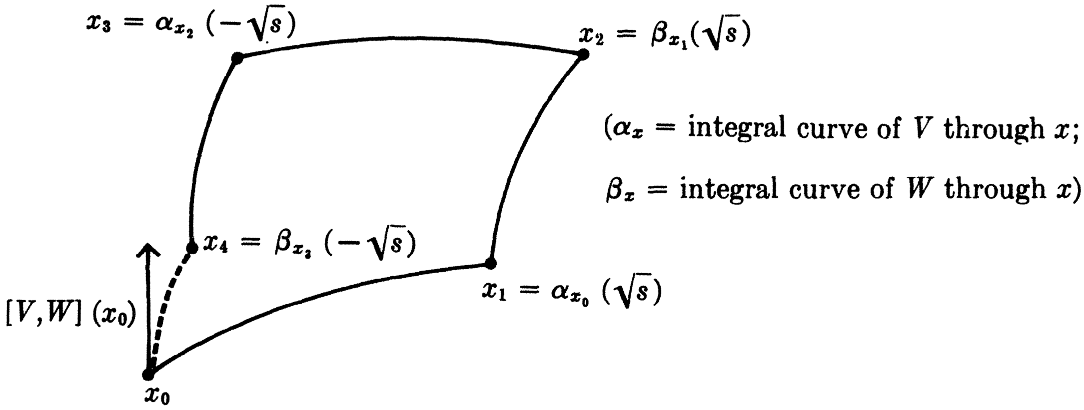


Figure 5.12

Definition Let V be an n-dimensional real vector space. Then $\Lambda^n(V^*)$ has dimension 1, so it is isomorphic to $\mathbb{R}^1$. Thus $\Lambda^n(V^*)\setminus\{0\}$ is disconnected; it is the union of two connected components. An *orientation of V* is a choice of one of these components. An *oriented vector space* is a pair $(V, \mathscr{A})$ where $\mathscr{A}$ is an orientation of V.

Remarks Thus each vector space V has two possible orientations. An ordered basis $\{\varphi_1, \ldots, \varphi_n\}$ of V^* determines an orientation of V; namely, the component of $\Lambda^n(V^*)$ in which $\varphi_1 \wedge \cdots \wedge \varphi_n$ lies. Given two ordered bases $\{\varphi_1, \ldots, \varphi_n\}$ and $\{\varphi_1', \ldots, \varphi_n'\}$ of V^*, with $\varphi_i' = \sum c_{ji}\varphi_j$, then

$$\varphi_1' \wedge \cdots \wedge \varphi_n' = \det(c_{ij})\, \varphi_1 \wedge \cdots \wedge \varphi_n$$

Hence two ordered bases determine the same orientation if and only if the determinant of the change of basis matrix is positive. In particular, if $\{\varphi_1, \ldots, \varphi_n\}$ is an ordered basis for V^*, then the orientation determined by the basis is different from the one determined by

$$\{\varphi_2, \varphi_1, \varphi_3, \ldots, \varphi_n\}$$

In $\mathbb{R}^2$, an orientation amounts to a sense of rotation. The orientation determined by $\{dr_1, dr_2\}$ gives the usual sense of positive rotation on $\mathbb{R}^2$; namely, so that the rotation sending $\partial/\partial r_1$ into $\partial/\partial r_2$ is one of $+\pi/2$. The orientation determined by $\{dr_2, dr_1\}$ defines the opposite sense of rotation, so that $\partial/\partial r_2 \to \partial/\partial r_1$ is a rotation of $+\pi/2$. (see Figure 5.13). Similarly, an orientation of $\mathbb{R}^3$ amounts to choosing either the right-handed rule or the left-handed rule for cross products.

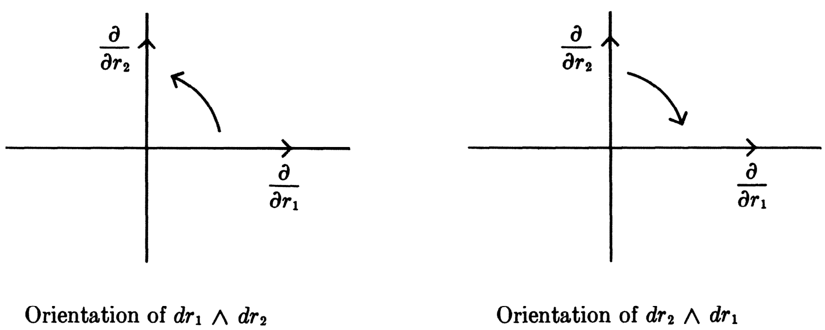


Figure 5.13

Definition A smooth manifold (X, Φ) is *orientable* if there exists a subset $\Phi' \subset \Phi$ such that

(1) $\{\text{domain}\,\varphi\}_{\varphi\in\Phi}$ is a covering of X, and

(2) If φ_1 and φ_2 are coordinate systems in Φ' with domains U and V and coordinate functions $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively, then the function $\lambda : U \cap V \to \mathbb{R}^1$ determined by

$$dx_1 \wedge \cdots \wedge dx_n = \lambda\, dy_1 \wedge \cdots \wedge dy_n \quad \text{is everywhere positive.}$$

An *orientation* of an orientable manifold (X, Φ) is a choice of subset $\Phi' \subset \Phi$ satisfying (1) and (2) and maximal with respect to (2). An *oriented manifold* is a triple (X, Φ, Φ') where(X, Φ) is an orientable manifold and Φ' is an orientation of (X, Φ).

Remark The function λ determined by $dx_1 \wedge \cdots \wedge dx_n = \lambda\, dy_1 \wedge \cdots \wedge dy_n$ is just the Jacobian determinant of $\varphi_1 \circ \varphi_2^{-1}$; that is,

$$\lambda = \det\left(\frac{\partial}{\partial y_j}(x_i)\right) = \det d\left(\varphi_1 \circ \varphi_2^{-1}\right).$$

In view ofthis, it is easy to check that a connected orientable manifold (X, Φ) has exactly two orientations Φ' and Φ'', and that Φ is the disjoint union $\Phi' \cup \Phi''$.

Remark A more sophisticated approach to orientation of manifolds is to consider the set $\Lambda^n(X)$. This set can be given the structure of an $(n+1)$-dimensional manifold as follows. Let $\varphi : U \to \mathbb{R}^n$ be a local coordinate system on X, with coordinate functions $(x_1, \ldots, x_n)$. Then a coordinate system $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{n+1}$ is defined on $\pi^{-1}(U)$ by

$$\tilde{\varphi}(\omega) = (\varphi(\pi(\omega)), \lambda(\omega)) \quad \text{for each } \omega \in \pi^{-1}(U),$$

where $\lambda : \pi^{-1}(U) \to \mathbb{R}^1$ is the function such that

$$\lambda(\omega)\, dx_1 \wedge \cdots \wedge dx_n = \omega \quad \text{for each } \omega \in \pi^{-1}(U).$$

In terms of $\Lambda^n(X)$, we have the following characterization of orientability.

Theorem 8 Let X be a connected smooth manifold (see Figure 5.14). Let

$$O = \bigcup_{x \in X} \{0 \text{ element in } \Lambda^n(T^*(X, x))\} \subset \Lambda^n(X).$$

Then either $\Lambda^n(X) \setminus O$ is connected, in which case X is not orientable, or $\Lambda^n(X) \setminus O$ breaks up into exactly two connected components, in which case X is orientable. An orientation of an orientable manifold X amounts to a choice of one of these two components.

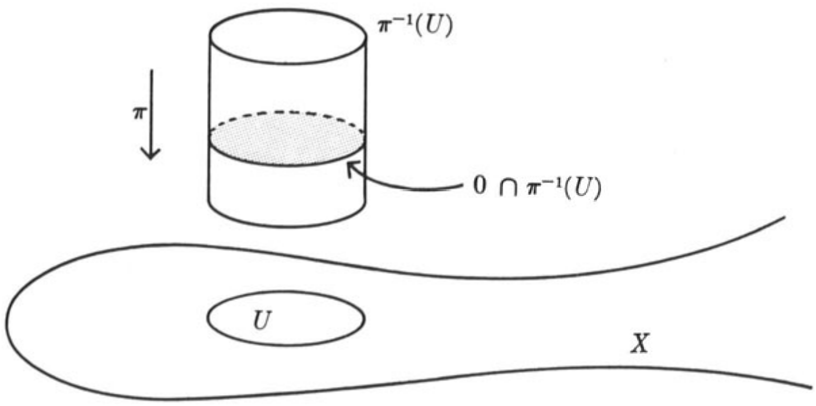


Figure 5.14

Proof We omit the proof.

Theorem 9 Let (X, Φ) be a smooth manifold of dimension n. Suppose there exists a smooth n-form ω on X which is nowhere zero. Then X is orientable.

Proof Let $\varphi \in \Phi$ be a local coordinate system on X, with connected domain U and coordinate functions $(x_1, \ldots, x_n)$. Since $\dim \Lambda^n(X) = 1$, there exists a smooth function $f_\varphi : U \to \mathbb{R}^1$ such that

$$\omega = f_\varphi \, dx_1 \wedge \cdots \wedge dx_n$$

Since ω is never zero, neither is f_φ. Thus either $f_\varphi > 0$ everywhere, or $f_\varphi < 0$ everywhere. Let

$$\Phi' = \{\varphi \in \Phi \mid f_\varphi > 0\}.$$

Then Φ' is an orientation of X. Φ' covers X because if $x \in X$ and φ is a coordinate system about x with $f_\varphi < 0$, then the new coordinate system $\tilde{\varphi}$ about x, obtained by changing the sign of one of the coordinate functions of φ, has $f_{\tilde{\varphi}} > 0$. Furthermore, if $\varphi, \psi \in \Phi'$ have domains U and V and coordinate functions $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively, then on $U \cap V$

$$dy_1 \wedge \cdots \wedge dy_n = \frac{1}{f_\psi} \omega = \frac{f_\varphi}{f_\psi} dx_1 \wedge \cdots \wedge dx_n$$

and $f_\varphi / f_\psi > 0$. Maximality is clear.

Theorem 10 Let (X, ψ) be an n-dimensional submanifold of$\mathbb{R}^{n+1}$. Suppose (X, ψ) admits a nonzero "normal vector field"; that is, suppose there exists a smooth map $V : X \to T(\mathbb{R}^{n+1})$ such that for each $x \in X$, $V(x)$ is a nonzero vector in $T(\mathbb{R}^{n+1}, \psi(x))$ perpendicular to $d\psi(T(X, x))$ (see Figure 5.15). Then X is orientable.

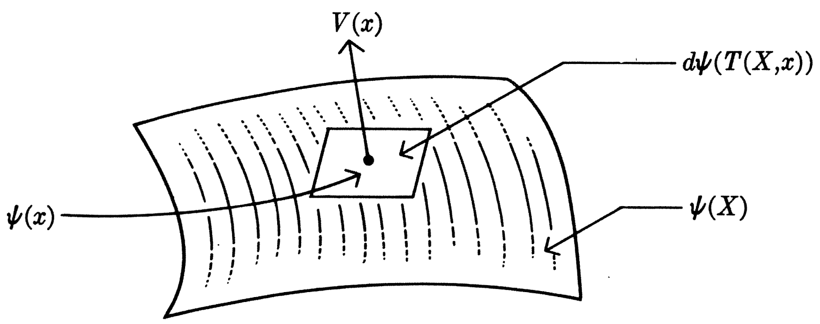


Figure 5.15

Remark Perpendicularity in $T(\mathbb{R}^{n+1}, \psi(x))$ means with respect to the inner product $\langle\, , \rangle$ given by

$$\langle \frac{\partial}{\partial r_i}, \frac{\partial}{\partial r_j} \rangle = \delta_{ij}.$$

Proof of Theorem 10 Given a normal vector field V, consider the n-form μ defined at points of $\psi(X)$ by

$$\mu = i(V)\, dr_1 \wedge \cdots \wedge dr_{n+1}.$$

Let $\omega = \psi^* \mu$. Then ω is a smooth n-form on X. By Theorem 9, it suffices to show ω is never zero on X. Suppose it were; that is, suppose $\omega(x) = 0$ for some $x \in X$. Then for all $v_1, \ldots, v_n \in T(X, x)$

$$\begin{aligned} 0 &= \omega(x)\,(v_1, \ldots, v_n) \\ &= \psi^* \mu(x)\,(v_1, \ldots, v_n) \\ &= \mu\,(\psi(x))\,(d\psi(v_1), \ldots, d\psi(v_n)) \end{aligned}$$

Now each vector $w \in T(\mathbb{R}^{n+1}, \psi(x))$ is of the form $w = d\psi(v) + cV(x)$ for some $v \in T(X, x)$, $c \in \mathbb{R}^1$. Thus, for arbitrary vectors

$$w_i = d\psi(v_i) + c_i V(x) \quad \text{for each } 1 \leq i \leq n.$$

we have

$$\begin{aligned} \mu(\psi(x))(w_1, \ldots, w_n) &= \mu(\psi(x))(d\psi(v_1) + c_1 V(x), \ldots, d\psi(v_n) + c_n V(x)) \\ &= \mu(\psi(x))(d\psi(v_1), \ldots, d\psi(v_n)) \\ &\quad + \sum_{j=1}^{n} c_j \mu(\psi(x))(d\psi(v_1), \ldots, d\psi(v_{j-1}), V(x), d\psi(v_{j+1}), \ldots d\psi(v_n)) \end{aligned}$$

All other terms are zero since $V(x)$ appears twice as an argument, and μ is skew symmetric. Moreover, the first term vanishes by the above discussion, and each term of the sum is zero because

$$\mu(\psi(x))(\ldots, V, \ldots) = i(V)\, dr_1 \wedge \cdots \wedge dr_{n+1}(\ldots, V, \ldots) = dr_1 \wedge \cdots \wedge dr_{n+1}(V, \ldots, V, \ldots) = 0.$$

Since $w_1, \ldots, w_n \in T(\mathbb{R}^{n+1}, \psi(x))$ were arbitrary, this shows that $\mu(\psi(x)) = 0$. But

$$\begin{aligned} \mu &= i(V)\, dr_1 \wedge \cdots \wedge dr_{n+1} \\ &= \sum_{j=1}^{n} (-1)^{j-1} (V r_j)\, dr_1 \wedge \cdots \wedge dr_{j-1} \wedge dr_{j+1} \wedge \cdot \wedge dr_{n+1} \end{aligned}$$

Since $V(x) r_j \neq 0$for some $1 \leq j \leq n$, $\mu(\psi(x)) \neq 0$. This contradiction proves the theorem.

Corollary The unit sphere $\mathbb{S}^n$ is orientable.

Proof $\mathbb{S}^n$ admits a nonzero normal vector field, namely, the restriction to $\mathbb{S}^n$ of the unit vector field on $\mathbb{R}^{n+1} \setminus \{0\}$ pointing radially outward.

Remark It can be shown that every compact connected n-dimensional submanifold of $\mathbb{R}^{n+1}$ separates $\mathbb{R}^{n+1}$ into two connected pieces, one bounded and one unbounded. Thus every such submanifold admits a unit normal vector field (for example, the one pointing into the unbounded component), hence is orientable.

Remark A nonorientable 2-dimensional manifold is called a one-sided surface.

Example 1. The *Möbius strip* S, obtained from an open rectangular strip by giving the strip a half twist and glueing the ends, is nonorientable. Note that a nonzero normal vector field cannot exist on S, for if such a field varies continuously along the center line, it would have to point in the opposite direction after a full circuit.

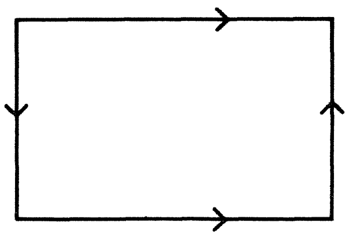


Figure 5.16

Example 2. The *Klein bottle* K, obtained from $I \times I$ by identifying opposite sides (to get a cylinder) and then identifying the other pair of sides with a twist (Figure 5.16), is nonorientable.

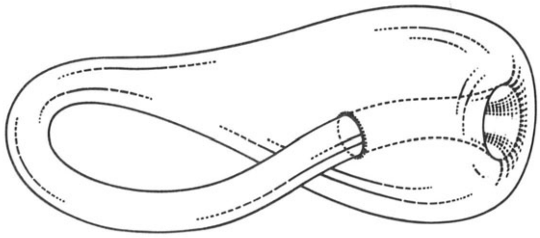


Figure 5.17

This surface cannot be represented as a submanifold of $\mathbb{R}^3$. However, there does exist a map $\psi : K \to \mathbb{R}^3$ with $d\psi$ injective at each point, and such that ψ is one-to-one except along a circle in $\mathbb{R}^3$ (see Figure 5.17).

Definition Let X be a topological space, and let $\mathscr{U}$ be an open covering of X. The covering $\mathscr{U}$ is *locally finite* if, for each $x \in X$, there exists an open set W_x containing x such that

$$\{U \in \mathscr{U} \mid U \cap W_x \neq \emptyset\} \quad \text{is a finite set.}$$

Definition A topological space X is *paracompact* if every open covering of X has a locally finite refinement; that is, if for every open covering $\mathscr{U}$, there exists a locally finite open covering $\mathscr{V}$ such that for each $V \in \mathscr{V}$ there exists a $U \in \mathscr{U}$ with $V \subset U$.

Remark It can be shown that all metric spaces are paracompact. Also, every regular topological (where) space whose topology has a countable basis is paracompact. Recall that a topological space X is a *regular space* if, given any closed subset F and any point $x \notin F$ of X, there exist disjoint open neighbourhoods U and V of x and F respectively.

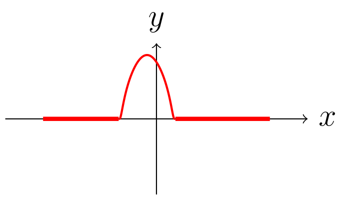
Partition of Unity

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

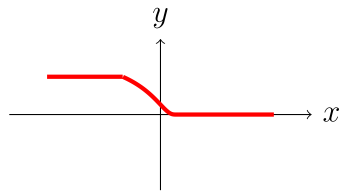
Then f is smooth, i.e. infinitely differentiable, on $\mathbb{R}$. For $a < b$, let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = f(x-a)f(b-x)$. Then g is smooth and

$$g(x) = \begin{cases} e^{-1/(x-a)}\, e^{-1/(b-x)} & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$



Let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(x) = \frac{\int_x^\infty g(t)\,dt}{\int_{-\infty}^\infty g(t)\,dt} \quad \text{for } x \in \mathbb{R}.$$



Then

$$\begin{cases} h(x) = 1 & \text{if } x \in (-\infty, a], \\ 0 < h(x) < 1 & \text{if } x \in (a, b), \\ h(x) = 0 & \text{if } x \in [b, \infty). \end{cases}$$

For each $p \in \mathbb{R}^n$ and for any $0 < r < t$, let $B_r(p)$ and $B_t(p)$ be concentric balls of radius r and t, respectively. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be the linear function such that $\zeta(r^2) = a$ and $\zeta(t^2) = b$, and let

$$\psi(x) = h(\zeta(\|x - p\|^2)) \quad \text{for } x \in \mathbb{R}^n.$$

Then $\psi : \mathbb{R}^n \to \mathbb{R}$ is smooth, $0 \le \psi(x) \le 1$ for all $x \in \mathbb{R}^n$ and

$$\psi(x) = \begin{cases} 1 & \text{if } x \in B_r(p), \\ 0 & \text{if } x \notin B_t(p). \end{cases}$$

Theorem Suppose K is a compact subset of $\mathbb{R}^n$, and $\{V_\alpha\}$ is an open cover of K. Then there exist functions $\psi_1, \dots, \psi_s \in C^\infty(\mathbb{R}^n)$, the space of smooth functions on $\mathbb{R}^n$, such that

(a) $0 \le \psi_i \le 1$ for $1 \le i \le s$;

(b) each ψ_i has its support in some V_α, i.e. $\overline{\{x \in \mathbb{R}^n \mid \psi_i(x) \neq 0\}} \subset V_\alpha$, and

(c) $\displaystyle\sum_{i=1}^{s} \psi_i(x) = 1$ for every $x \in K$.

Because of (c), $\{\psi\}$ is called a partition of unity, and (b) is sometimes expressed by saying that $\{\psi_i\}$ is subordinate to cover $\{V_\alpha\}$.

Corollary If $f \in \mathscr{C}(\mathbb{R}^n)$ is a continuous function in $\mathbb{R}^n$ and the support of f lies in K, then

$$f = \sum_{i=1}^{s} \psi_i f.$$

Each $\psi_i f$ has its support in some V_α.

Proof For each $x \in K$, since $\{V_\alpha\}$ is an open cover of K, there exist $V_{\alpha(x)} \in \{V_\alpha\}$, open balls $B(x)$ and $W(x)$, centered at x, such that

$$(*) \quad \overline{B(x)} \subset W(x) \subset \overline{W(x)} \subset V_{\alpha(x)}.$$

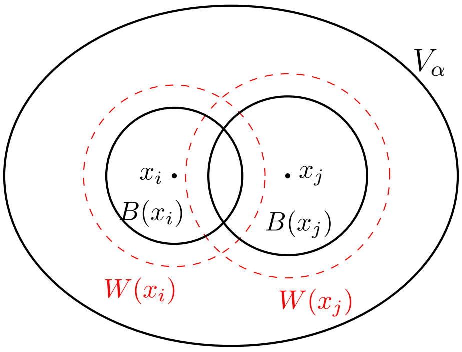
Since K is compact and $\{B(x) \mid x \in K\}$ is an open cover of K, there are points $x_1, \dots, x_s \in K$ such that

$$K \subset B(x_1) \cup \cdots \cup B(x_s).$$

By $(*)$, there are functions $\phi_1, \ldots, \phi_s \in \mathscr{C}(\mathbb{R}^n)$, such that

$$\phi_i(x) = \begin{cases} 1 & \text{if } x \in B(x_i) \\ 0 & \text{if } x \in \mathbb{R}^n \setminus W(x_i) \end{cases}$$

and $0 \le \phi_i(x) \le 1$ for all $x \in \mathbb{R}^n$ for each $1 \le i \le s$.



Define

$$\begin{aligned} \psi_1 &= \phi_1 \\ (\dagger) \quad \psi_{i+1} &= (1-\phi_1)\cdots(1-\phi_i)\phi_{i+1} \quad \text{for } i = 1, \ldots, s-1. \end{aligned}$$

Properties (a) and (b) are clear. The relation

$$(\dagger\dagger) \quad \psi_1 + \cdots + \psi_i = 1 - (1-\phi_1)\cdots(1-\phi_i)$$

is trivial for $i = 1$. If $(\dagger)$ holds for some $i < s$, addition of $(\dagger)$ and $(\dagger\dagger)$ yields $(\dagger\dagger)$ with $i+1$ in place of i. It follows that

$$\sum_{i=1}^{s} \psi_i(x) = 1 - \prod_{i=1}^{s} \big[1 - \phi_i(x)\big] \quad \text{for } x \in \mathbb{R}^n.$$

If $x \in K$, then $x \in B(x_j)$ for some $1 \le j \le s$, hence $\phi_j(x) = 1$, $\prod_{i=1}^{s} \big[1 - \phi_i(x)\big] = 0$ and $\sum_{i=1}^{s} \psi_i(x) = 1$. This proves (c).

Theorem Let $A \subset \mathbb{R}^n$ and let $\mathscr{O}$ be a collection of open subsets of $\mathbb{R}^n$ covering A. Then there is a collection Φ of continuous functions φ defined in an open set containing A, with the following properties:

(1) For each $x \in A$ we have $0 \le \varphi(x) \le 1$.

(2) For each $x \in A$ there is an open set V containing x such that all but finitely many of $\varphi \in \Phi$ are 0 on V.

(3) For each $x \in A$ we have $\sum_{\varphi \in \Phi} \varphi(x) = 1$ (by (2) for each x this sum is finite in some open set containing x).

(4) For each $\varphi \in \Phi$ there is an open set $U \in \mathscr{O}$ such that $\varphi = 0$ outside of some closed set contained in U.

A collection Φ satisfying (1) to (3) is called a continuous partition of unity for A. If Φ also satisfies (4), it is said to be subordinate to the cover $\mathscr{O}$. In this chapter we will only use continuity of the functions φ.

Proof

Case 1. A is compact.

We use an alternative method to construct a continuous partition of unity as follows.

Since $A \subset \mathbb{R}^n$ is compact, there exists an open ball $B_R(0)$ of radius R centered at 0 such that $A \subset B_R(0)$. By taking $U \cap B_R(0)$ for each $U \in \mathscr{O}$, we may assume that $\mathscr{O}$ is a collection of bounded open subsets covering A. Again since A is compact, there exist open sets $U_1, \ldots, U_m \in \mathscr{O}$ such that

$$(*) \quad A \subset \bigcup_{j=1}^{m} U_j \quad \text{and} \quad A \setminus \big(U_1 \cup \cdots \cup \widehat{U_i} \cup \cdots \cup U_m\big) \neq \emptyset \quad \forall\, 1 \leq i \leq m,$$

where $\widehat{U_i}$ means the term U_i is omitted.

Since A is compact and by $(*)$, the set $C_1 = A \setminus \bigcup_{j=2}^{m} U_j$ is a compact subset of U_1 with

$$r_1 = d(\partial U_1, C_1) = \inf_{x \in \partial U_1,\, y \in C_1} d(x, y) > 0.$$

Let

$$\begin{aligned}
D_1 &= \{x \in U_1 \mid d(x, \partial U_1) = \inf_{y \in \partial U_1} d(x, y) \geq r_1/2\}, \\
W_1 &= \{x \in U_1 \mid d(x, \partial U_1) = \inf_{y \in \partial U_1} d(x, y) \geq r_1/4\}, \\
\psi_1(x) &= \begin{cases} 1 & \text{if } x \in D_1, \\ 0 & \text{if } x \notin W_1, \\ 0 \leq \psi_1(x) \leq 1 & \forall\, x \in \mathbb{R}^n. \end{cases}
\end{aligned}$$

Note that D_1 is a compact subset of U_1, $C_1 \subset \operatorname{Int} D_1$ and $A \subset \big(\operatorname{Int} D_1\big) \cup \bigcup_{j=2}^{m} U_j$.

Suppose that $D_1, \ldots, D_k$ have been chosen so that $A \subset \big(\bigcup_{j=1}^{k} \operatorname{Int} D_j\big) \cup \big(\bigcup_{j=k+1}^{m} U_j\big)$. Let

$$C_{k+1} = A \setminus (\operatorname{Int} D_1 \cup \cdots \cup \operatorname{Int} D_k \cup U_{k+2} \cup \cdots \cup U_m).$$

Then $C_{k+1} \subset U_{k+1}$ is a compact with

$$r_{k+1} = d(\partial U_{k+1}, C_{k+1}) = \inf_{x \in \partial U_{k+1},\, y \in C_{k+1}} d(x, y) > 0.$$

Let

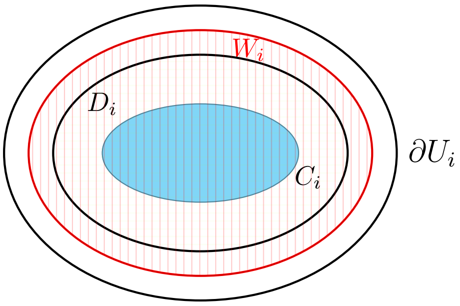
$$\begin{aligned} D_{k+1} &= \{x \in U_{k+1} \mid d(x, \partial U_{k+1}) = \inf_{y \in \partial U_{k+1}} d(x,y) \geq r_{k+1}/2\}, \\ W_{k+1} &= \{x \in U_1 \mid d(x, \partial U_1) = \inf_{y \in \partial U_1} d(x,y) \geq r_{k+1}/4\}, \\ \psi_{k+1}(x) &= \begin{cases} 1 & \text{if } x \in D_{k+1}, \\ 0 & \text{if } x \notin W_{k+1}, \\ 0 \leq \psi_{k+1} \leq 1 & \forall\, x \in \mathbb{R}^n. \end{cases} \end{aligned}$$

Note that D_{k+1} is a compact subset of U_{k+1}, $C_{k+1} \subset \operatorname{Int} D_{k+1}$ and $A \subset \bigcup_{j=1}^{k+1} \operatorname{Int} D_j \cup \bigcup_{j=k+2}^{m} U_j$.

We obtain a collection of compact subsets $\{D_i\}_{i=1}^m$, $\{W_i\}_{i=1}^m$ and a collection of nonnegative continuous functions $\{\psi_i\}_{i=1}^m$ such that

$$A \subset \bigcup_{i=1}^m \operatorname{Int} D_i \subset \bigcup_{i=1}^m D_i \subset \bigcup_{i=1}^m \operatorname{Int} W_i \subset \bigcup_{i=1}^m W_i \subset \bigcup_{i=1}^m U_i,$$

$$\begin{cases} \psi_i(x) = 1 & \text{if } x \in D_i, \\ \psi_i(x) = 0 & \text{if } x \notin W_i. \end{cases} \implies \sum_{j=1}^m \psi_j(x) > 0 \quad \text{for all } x \in \bigcup_{i=1}^m D_i.$$



For each $1 \leq i \leq m$, and for $x \in \bigcup_{i=1}^m D_i$, if we let

$$\varphi_i(x) = \frac{\psi_i(x)}{\sum_{j=1}^m \psi_j(x)},$$

then $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ is the desired partition of unity since $\{D_1, \ldots, D_m\}$ covers A.

Case 2. $A = A_1 \cup A_2 \cup A_3 \cup \cdots$, where each A_i is compact and $A_i \subset \operatorname{Int} A_{i+1}$.

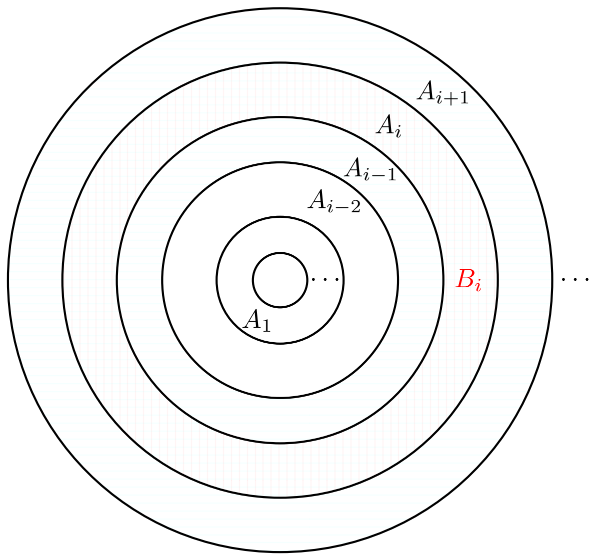
For each $i \in \mathbb{N}$, let

$$B_i = \begin{cases} A_1 & \text{if } i = 1, \\ A_i \setminus \operatorname{Int} A_{i-1} & \text{if } i \geq 2, \end{cases}$$

and

$$\mathscr{O}_i = \begin{cases} \{U \cap \operatorname{Int} A_3 \mid U \in \mathscr{O}\} & \text{if } 1 \leq i \leq 2, \\ \{U \cap (\operatorname{Int} A_{i+1} \setminus A_{i-2}) \mid U \in \mathscr{O}\} & \text{if } i \geq 3. \end{cases}$$

Then



(i) each B_i is compact and $A = \bigcup_{i=1}^{\infty} B_i$,

(ii) each $\mathscr{O}_i$ is a collection of bounded open sets covering B_i, i.e. $B_i \subset \bigcup_{U \in \mathscr{O}_i} U$.

(iii) $U \cap V = \emptyset$ for all $U \in \mathscr{O}_i$, $V \in \mathscr{O}_j$ with $j > i+2 \iff j-1 > i+1$.

Thus, by case 1, there is a partition of unity Φ_i for B_i, subordinate to $\mathscr{O}_i$.

Note that for each $x \in A$, since $x \in B_i$ for some B_i and since Φ_i is a partition of unity subordinate to $\mathscr{O}_i$, so there exists $\varphi \in \Phi_i$ such that $\varphi(x) > 0$, and by (iii),

$$\varphi(x) = 0 \quad \forall \varphi \in \Phi_j \text{ with } j > i+2 \iff j-1 > i+1,$$

and the sum

$$\sigma(x) = \sum_{\varphi \in \Phi_i, \text{ all } i} \varphi(x)$$

is a finite sum in some open set containing x, and $\sigma(x) > 0$ for all $x \in A$.

Let

$$\Phi = \bigcup_{i=1}^{\infty} \left\{ \frac{\varphi(x)}{\sigma(x)} \mid \varphi \in \Phi_i \right\}.$$

Then Φ is a partition of unity subordinate to the open cover $\mathscr{O}$.

Case 3. A is open.

Let

$$A_i = \{x \in A \mid \|x\| \leq i \text{ and } d(x, \partial A) \geq \frac{1}{i}\},$$

where $d(x, \partial A) =$ the distance from x to the boundary ∂A. Note that A_i is compact, $A_i \subset \operatorname{Int} A_{i+1}$ for all $i \geq 1$, and

$$\bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i = A.$$

By applying the case 2, we obtain a partition of unity subordinate to the open cover $\mathscr{O}$.

Case 4. A is arbitrary.

Let B be the union of all U in $\mathscr{O}$. By case 3 there is a partition of unity for B; this is also a partition of unity for A.

Definition Let X be a smooth manifold. A smooth *partition of unity* on X is a pair $(\mathscr{V}, \mathscr{F})$ where $\mathscr{V}$ is a locally finite covering of X and $\mathscr{F} = \{f_V\}_{\mathscr{V}}$ is a collection of smooth real-valued functions on X such that

(1) $f_V(x) \geq 0$ for each $V \in \mathscr{V}$, $x \in X$,

(2) for each $V \in \mathscr{V}$, the support of $f_V =$ the closure of the set $\{x \in X \mid f_V(x) \neq 0\}$ is contained in V, i.e.
$$\operatorname{supp}(f_V) = \overline{\{x \in X \mid f_V(x) \neq 0\}} \subset V$$

(3) $\sum_{V \in \mathscr{V}} f_V(x) = 1$ for each $x \in X$. (Note that this sum makes sense since for each $x \in X$, $f_V(x) = 0$ for all but finitely many $V \in \mathscr{V}$.)

Theorem 11 Let X be a paracompact manifold. Then, given any open covering $\mathscr{U}$ of X, there exists a smooth partition of unity $\mathscr{F} = \{f_V\}_{\mathscr{V}}$ on X such that $\mathscr{V}$ is a refinement of $\mathscr{U}$.

Proof Since X is a manifold, there is a refinement $\mathscr{W}$ of $\mathscr{U}$ such that each open set $W \in \mathscr{W}$ is a coordinate neighborhood, and $\bar{W}$ is compact. Since X is paracompact, there is a locally finite refinement $\mathscr{V}$ of the open covering $\mathscr{W}$. Note that $\mathscr{V}$ is a refinement of $\mathscr{U}$, and if $V \in \mathscr{V}$, then $\bar{V}$ is compact, and V is a coordinate neighborhood.

Suppose we can "shrink the covering $\mathscr{V}$ slightly" and still get a covering. That is, suppose for each $V \in \mathscr{V}$, we can choose an open set $\alpha(V)$ such that $\overline{\alpha(V)} \subset V$ and $\{\alpha(V)\}_{V \in \mathscr{V}}$ is a covering. We then proceed as follows. Since $V \in \mathscr{V}$ is a coordinate neighborhood, and $\overline{\alpha(V)}$ is a compact set in V, we can find a smooth nonnegative function $g_V : X \to \mathbb{R}^1$ such that

$$g_V(x) = \begin{cases} 1 & \text{if } x \in \alpha(V), \\ 0 & \text{if } x \notin V. \end{cases}$$

Let $g = \sum_{V \in \mathscr{V}} g_V$. Then g is well defined and in $C^\infty(X, \mathbb{R}^1)$ because $\mathscr{V}$ is locally finite. Furthermore, g never vanishes on X because $\{\alpha(V)\}_{V \in \mathscr{V}}$ is a covering; hence $f_V = g_V/g \in C^\infty(X, \mathbb{R}^1)$. Let $\mathscr{F} = \{f_V\}_{V \in \mathscr{V}}$; then $(\mathscr{V}, \mathscr{F})$ is a smooth partition of unity.

To "shrink the covering $\mathscr{V}$ slightly," proceed as follows. Consider the family $\mathscr{B}$ of all functions β such that

(1) domain of β is a subset $\mathscr{D}_\beta$ of $\mathscr{V}$;

(2) if $V \in \mathscr{D}_\beta$, then $\beta(V)$ is an open set in V such that $\overline{\beta(V)} \subset V$; and

(3) the collection of open sets $\{\beta(V) \mid V \in \mathscr{D}_\beta\} \cup \{V \mid V \notin \mathscr{D}_\beta\}$ is an open covering of X.

The family $\mathscr{B}$ is partially ordered:

$$\beta < \gamma \text{ if } \mathscr{D}_\beta \subset \mathscr{D}_\gamma \text{ and } V \in \mathscr{D}_\beta \implies \beta(V) = \gamma(V).$$

We leave the following point set argument to the reader: since $\mathscr{V}$ is locally finite, the maximum principle implies that $\mathscr{B}$ has a maximal element α and $\mathscr{D}_\alpha = \mathscr{V}$, so that α is the required shrinkage.

Theorem 12 Let X be a paracompact manifold that is orientable. Then there exists a smooth n-form ω on X such that ω never vanishes.

Proof Let Φ' be an orientation of X. Let $\mathscr{U} = \{\text{domain}\,\varphi\}_{\varphi\in\Phi'}$. Then $\mathscr{U}$ is an open covering of X. Let $(\mathscr{V}, \mathscr{F})$ be a smooth partition of unity such that $\mathscr{V}$ is a refinement of $\mathscr{U}$. For each $V \in \mathscr{V}$, let $\varphi_V \in \Phi'$ be such that $V \subset \text{domain}\,\varphi_V$. Then the restriction of φ_V to V is also an element of Φ'. Let $\left(x_1^V, \ldots, x_n^V\right)$ denote the coordinate functions on V. Then the n-form $\omega^V = dx_1^V \wedge \cdots \wedge dx_n^V \in C^\infty(V, \Lambda^n(V))$ is nowhere zero on V. Let $\omega = \displaystyle\sum_{V\in\mathscr{V}} f_V\omega^V$, where $f_V\omega^V$ is by definition zero outside V. Then $\omega \in C^\infty(X, \Lambda^n(X))$.

We must show that ω is nowhere zero. For $x \in X$, let $\varphi \in \Phi'$ be a coordinate system about x, with domain U and coordinate functions $(y_1, \ldots, y_n)$. Then, for each $V \in \mathscr{V}$ with $U \cap V \neq \emptyset$,

$$\omega^V = dx_1^V \wedge \cdots \wedge dx_n^V = g_V\, dy_1^V \wedge \cdots \wedge dy_n^V \quad \text{on } U \cap V,$$

and $g_V > 0$ on $U \cap V$ since both φ_V and φ are members of Φ'. Thus,

$$\omega|_U = \sum_{V\in\mathscr{V}} \left(f_V\omega^V\right)|_U = \left(\sum_{V\in\mathscr{V}} f_V g_V\right) dy_1^V \wedge \cdots \wedge dy_n^V.$$

Since $\displaystyle\sum_{V\in\mathscr{V}} f_V = 1$, there exists $V_0 \in \mathscr{V}$ such that $f_{V_0}(x) > 0$. Since $g_{V_0}(x) \neq 0$ and each $f_V g_V \geq 0$, $\left(\displaystyle\sum_{V\in\mathscr{V}} f_V g_V\right)(x) \neq 0$ and $\omega(x) \neq 0$.

Remark Theorems 9 and 12 completely characterize orientability of paracompact manifolds by the existence or nonexistence of a nonzero n-form. This characterization can be applied to show that the projective space P^n is orientable if and only if n is odd. This is done by considering the sphere S^n as a covering space of P^n with covering map p. Let ω be the nonzero n-form on S^n constructed in the proof of Theorem 10 and its corollary. Then one can show that for n odd, ω defines an n-form $\tilde{\omega}$ on P^n such that $\omega = p^*\tilde{\omega}$. If P^n were orientable for n even, then there would exist a nonzero n-form $\tilde{\omega}$ on P^n and then $p^*\tilde{\omega} = g\omega$ for some $g \neq 0$. On the other hand, one can check that if $x_1 \neq x_2 \in S^n$ are such that $p(x_1) = p(x_2)$, then $g(x_1) > 0 \iff g(x_2) < 0$, contradicting the fact that g is never zero.

Remark A nonzero smooth n-form on a smooth n-manifold is called a *volume* element. Thus every orientable paracompact manifold admits a volume element. The form $i(V)\, dr_1 \wedge \cdots \wedge dr_{n+1}$ on S^n discussed in Theorem 10 and its corollary is the usual volume element on the n-sphere.

Definition A *Riemannian manifold* is a smooth manifold X,together with a map

$$\langle\, ,\, \rangle : X \to \bigcup_{x\in X} \{\text{inner product on } T(X, x)\}$$

such that for each $x \in X$, $\langle\, ,\, \rangle(x)$ (usually denoted $\langle\, ,\, \rangle_x$) is an inner product on $T(X, x)$ and such that $\langle\, ,\, \rangle$ is smooth; that is, for each pair V_1, V_2 of smooth vector fields on X, $\langle V_1, V_2\rangle$ is a smooth function, where

$$\langle V_1, V_2\rangle(x) = \langle V_1(x), V_2(x)\rangle_x \quad \text{for } x \in X.$$

The map $\langle V_1, V_2\rangle$ is called a *Riemannian structure* on X.

Theorem 13 Let X be a paracompact smooth manifold. Then there exists a Riemannian structure on X.

Proof Let $(\mathscr{V}, \mathscr{F})$ be a smooth partition of unity on X such that each $V \in \mathscr{V}$ is a coordinate neighborhood. Define a Riemannian structure $\langle\,,\,\rangle_V$ on each $V \in \mathscr{V}$ by

$$\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \rangle_V = \delta_{ij}, \quad 1 \le i, j \le n,$$

where $(x_1, \ldots, x_n)$ are the coordinate functions on V. Then define $\langle\,,\,\rangle$ on X by

$$\langle\,,\,\rangle = \sum_{V \in \mathscr{V}} f_V \langle\,,\,\rangle_V.$$

Remark The converse of Theorem 13 also holds; namely, every Riemannian manifold is paracompact.

Example 1. $\mathbb{R}^n$ is a Riemannian manifold: take $\{\partial/\partial r_i\}$ as an orthonormal basis for the tangent space at each point.

Example 2. Let X be a Riemannian manifold, and let (Y, i) be a submanifold of X. Then a Riemannian structure is given on Y by

$$\langle v_1, v_2 \rangle_y = \langle di(v_1), di(v_2) \rangle_{i(y)} \quad \text{for all } v_1, v_2 \in T(Y, y).$$

Example 3. In view of Example 2, every submanifold of $\mathbb{R}^n$ has a Riemannian structure.

Example 4. Let X and Y be Riemannian manifolds. Then the manifold $X \times Y$ has a Riemannian structure given as follows. For $(x, y) \in X \times Y$, the tangent space $T(X \times Y, (x, y))$ is naturally isomorphic to the direct sum of the vector spaces $T(X, x)$ and $T(Y, y)$. An inner product on $T(X \times Y, (x, y))$ is then given by requiring that this isomorphism be an isometry with the orthogonal direct sum $T(X, x) \oplus T(Y, y)$.

Definition Let X and Y be Riemannian manifolds. A map $\varphi : X \to Y$ is an *isometry* if it is smooth, injective, surjective, has a smooth inverse, and is such that $d\varphi$ is an isometry at each point; that is,

$$\langle d\varphi(v_1), d\varphi(v_2) \rangle_{\varphi(x)} = \langle v_1, v_2 \rangle_x \quad \text{for all } v_1, v_2 \in T(X, x) \text{ and } x \in X.$$

Remark Thus an isometry preserves all the structure of a Riemannian manifold. Two manifolds are equivalent from the viewpoint of Riemannian geometry if there exists an isometry between them. Such manifolds are said to be *isometric.* Note that two Riemannian manifolds as smooth manifolds can be the same; yet as Riemannian manifolds, be distinct.

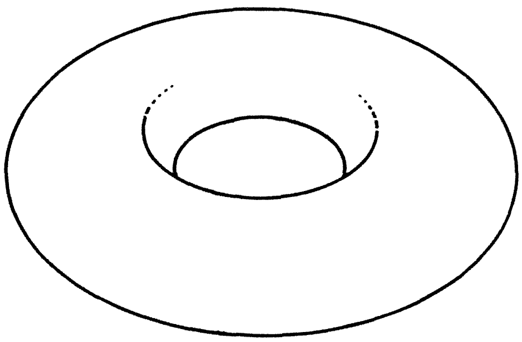


Figure 5.18

Example 5. Consider the torus $S^1 \times S^1$. It has a Riemannian structure as a submanifold of $\mathbb{R}^3$ (see Figure 5.18). On the other hand, it has a Riemannian structure as a product $S^1 \times S^1$, where S^1 is given a Riemannian structure by way of its usual imbedding into $\mathbb{R}^2$. These two structures are distinct. In fact, the product structure on $S^1 \times S^1$ cannot be obtained by representing $S^1 \times S^1$ as a submanifold of $\mathbb{R}^3$. (see Chapter 8). However, it can be obtained as a submanifold of $\mathbb{R}^4$ since

$$S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4.$$